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A COMBINED HELMHOLTZ INTEGRAL EQUATION – FOURIER SERIES FORMULATION OF ACOUSTACAL RADIATION AND SCATERING PROBLEMS

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Abstract

The Combined Helmholtz Integral Equation – Fourier series Formulation (CHIEFF) is based on representation of a velocity potential in terms of Fourier series and finding the Fourier coefficients of this expansion. The solution could be substantially simplified if the Green function and its normal derivative are represented by Fourier series. Unfortunately the direct expansion of the Green function and its normal derivative are impossible because these functions do not satisfy the Dirichlet's theorem due to the singularities. To take advantage of the Fourier methods it is necessary to reformulate the original Helmholtz integral equation so that the modified Green function does not contain any singularities. The corresponding revision of the problem is proposed in the present paper. The Green function is modified so to satisfy the Dirichlet's theorem. The tradeoff is that the original Helmholtz integral equation contains new double singular integrals which could be calculated numerically by an adaptive procedure or by means of quadrature formulae. Fourier coefficients of the modified Green functions are calculated using a discrete Fourier transform, in particular case by FFT. Using orthogonality of the sine and cosine functions the original problem is reduced to an overdetermined system of linear algebraic equations to obtain the unknown coefficients of the Fourier series expansion. The CHIEFF method is applicable to a broad range of acoustical problems of radiation and scattering. It is especially effective for calculation of near acoustic fields of large-scale structures.

INTRODUCTION

Combined Helmholtz integral equation formulation is often used in boundary element methods for description of the effects of radiation and scattering of physical fields, for example, acoustical fields. This method has a substantial advantage over the "domain" methods, such as finite element methods, due to reduction of three dimensional problems to two dimensions. In his paper Seybert [1] described the method of integral equations for solution of radiation and scattering problems for axisymmetric bodies and boundary conditions. Further Soenarko [2] generalized this method to the problems with axisymmetric bodies and arbitrary boundary conditions. Present paper considers a general case of arbitrary body, which could be uniquely characterized by a system of two parameters, with arbitrary boundary conditions. After formulation of idea of the method the algorithm is formulated and numerical example of a plane acoustic wave scattering by a cylindrical body with spherical end caps is considered.

IDEA OF THE METHOD

Helmholtz integral equation for the radiation-scattering steady-state acoustic field could be written as follows:

$$C(P)\cdot\Phi(P)=\int_{(A)}\left[\frac{\partial\ g(P,Q)}{\partial n_Q}\cdot\Phi(Q)+g(P,Q)\cdot V_n(Q)\right]dA+4\pi\cdot\Psi^{(i)}(P) \tag{1}$$

where $\Phi(P)$ - velocity potential of the acoustic field at point P; $\Phi(Q)$ - velocity potential of the acoustic field at point Q supposing that this point belongs to the surface area (A); $\Psi^{(i)}(P)$ - velocity potential of an incident acoustic field at point P; $V_n(Q)$ - normal component of linear velocity of the surface (A) at point Q; $g(P,Q)=\dfrac{e^{-i\,k\,\rho(P,Q)}}{\rho(P,Q)}$ - Green function of the problem, $k=\omega/c$ - wave number; $\rho(P,Q)$ - distance between points P and Q; $\dfrac{\partial\ g(P,Q)}{\partial n_Q}$ - normal outer derivative of the Green function; $C(P)$ - coefficient, depending on location of point P, which is equal to 4π, if P is in outer space of the surface; 0, if P is inside the surface. In general case

$$C(P)=4\pi+\int_{(A)}\frac{\partial}{\partial n_Q}\left[\frac{1}{\rho(P,Q)}\right]dA \tag{2}$$

This expression equals to 2π for a smooth surface.

The main idea of our method is that we change the Green function and its normal derivative in a small vicinity of point P and hence, eliminate their singularities, for example, suppose these functions are equal to zero. The modified functions satisfy the Dirichlet's theorem and could be expanded in Fourier series.

Residual parts of integral expressions in (1) are approximately estimated in the vicinity of point P. Following this method we rewrite equation (1) as follows:

$$C(P)\cdot\Phi(P)=\int\limits_{(A)}\left[\frac{\partial\ g(P,Q)}{\partial n_Q}\cdot\Phi(Q)+g(P,Q)\cdot V_n(Q)\right]dA+4\pi\cdot\Psi^{(i)}(P)\approx$$
$$\approx\int\limits_{(A)}\left[\widetilde{\frac{\partial\ g(P,Q)}{\partial n_Q}}\cdot\Phi(Q)+\widetilde{g(P,Q)}\cdot V_n(Q)\right]dA+ \tag{3}$$
$$\Phi(P)\cdot\int\limits_{(\Delta A)}\frac{\partial\ g(P,Q)}{\partial n_Q}d(\Delta A)+V_n(P)\cdot\int\limits_{(\Delta A)}g(P,Q)\,d(\Delta A)+4\pi\cdot\Psi^{(i)}(P)$$

where $\widetilde{g(P,Q)}$- modified Green function, which is equal to $g(P,Q)$ for all points Q outside a small area ΔA, surrounding point P, and 0 at $Q\in\Delta A$; $\widetilde{\frac{\partial\ g(P,Q)}{\partial n_Q}}$ - modified normal derivative of Green function, which is equal to $\frac{\partial\ g(P,Q)}{\partial n_Q}$ for all points Q outside a small area ΔA, surrounding point P, and 0 at $Q\in\Delta A$.

Equation (3) could be rewritten as follows:

$$\widetilde{C(P)}\cdot\Phi(P)\approx\int\limits_{(A)}\left[\widetilde{\frac{\partial\ g(P,Q)}{\partial n_Q}}\cdot\Phi(Q)+\widetilde{g(P,Q)}\cdot V_n(Q)\right]dA+V_n(P)\cdot\int\limits_{(\Delta A)}g(P,Q)\,d(\Delta A)+4\pi\cdot\Psi^{(i)}(P) \tag{4}$$

where (see (2))

$$\widetilde{C(P)}=4\pi+\int\limits_{(A)}\frac{\partial}{\partial n_Q}\left[\frac{1}{\rho(P,Q)}\right]dA-\int\limits_{(\Delta A)}\frac{\partial\ g(P,Q)}{\partial n_Q}d(\Delta A) \tag{5}$$

In the case of a radiation–scattering problem with a prescribed value of normal projection of surface velocity ($V_n(Q)$) equation (4) is:

$$\widetilde{C(P)}\cdot\Phi(P)\approx\int\limits_{(A)}\widetilde{\frac{\partial\ g(P,Q)}{\partial n_Q}}\cdot\Phi(Q)\,dA+\left\{\int\limits_{(A)}\widetilde{g(P,Q)}\cdot V_n(Q)\,dA+V_n(P)\cdot\int\limits_{(\Delta A)}g(P,Q)\,d(\Delta A)+4\pi\cdot\Psi^{(i)}(P)\right\} \tag{6}$$

where the expression in braces could be considered as known function.

It is recommended to calculate the integrals $\int\limits_{(\Delta A)}\frac{\partial\ g(P,Q)}{\partial n_Q}d(\Delta A)$, $\int\limits_{(\Delta A)}g(P,Q)\,d(\Delta A)$ numerically using the adaptive algorithm.

HELMHOLTZ INTEGRAL EQUATION FORMULATION IN SPHERICAL COORDINATES

Suppose that our surface is so that it is possible to make a one-to-one correspondence between the surface points and the spherical angular coordinates (θ,φ), i.e. that a radius vector exists $r = r(\theta,\varphi)$, which realizes the isomorphism between points of the surface (A) and coordinates (θ,φ). In this case calculations of surface integrals in (5)-(6) could be carry out by

$$\int\limits_{(A)} \Psi\left(\theta_Q,\varphi_Q\right) dA = \int\limits_0^{2\pi}\int\limits_0^{\pi} \Psi\left(\theta_Q,\varphi_Q\right)\cdot\frac{r^2\left(\theta_Q,\varphi_Q\right)}{\cos\left[\overrightarrow{r\left(\theta_Q,\varphi_Q\right)},\overrightarrow{n_Q}\right]}\cdot\sin\theta_Q\, d\theta_Q\, d\varphi_Q \qquad (7)$$

where $\cos\left[\overrightarrow{r\left(\theta_Q,\varphi_Q\right)},\overrightarrow{n_Q}\right]$ - cosine of an angle between the radius vector $\overrightarrow{r\left(\theta_Q,\varphi_Q\right)}$, drawn from the centre of the spherical coordinates till point Q on the surface (A) and the outer normal $\overrightarrow{n_Q}$ drawn to the surface (A) at point Q.

Let us seek a solution in the form of a truncated Fourier series:

$$\Phi(\theta,\varphi) \approx a_0 + \sum_{n=1}^{N}\sum_{m=1}^{M}\left[a_{mn}\cos(m\varphi) + b_{mn}\sin(m\varphi)\right]\cdot\cos(n\theta) \qquad (8)$$

In expression (16) $2\cdot M\cdot N+1$ coefficients a_0, a_{mn}, b_{mn} , $(m=1,2,\ldots,M;\, n=1,2,\ldots,N)$ are unknown and does not depend on a particular point P. To find them we will use the orthogonality of Fourier series. To do this we have to expand

$$f_1\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right) = \frac{\overline{\partial\; g\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right)}}{\partial n_Q}\cdot\frac{r^2\left(\theta_Q,\varphi_Q\right)}{\cos\left[\overrightarrow{r\left(\theta_Q,\varphi_Q\right)},\overrightarrow{n_Q}\right]}\cdot\sin\theta_Q \quad \text{and}$$

$$f_2\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right) = \overline{g\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right)}\cdot\frac{r^2\left(\theta_Q,\varphi_Q\right)}{\cos\left[\overrightarrow{r\left(\theta_Q,\varphi_Q\right)},\overrightarrow{n_Q}\right]}\cdot\sin\theta_Q$$

in Fourier series, where $\overline{g\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right)} = \widetilde{g(P,Q)}$. The main difficulty is that both $f_1\left(\theta_Q,\varphi_Q\right)$ and $f_2\left(\theta_Q,\varphi_Q\right)$ are defined on the interval $\theta_Q \in [0,\pi]$ and we need to redefine it on the interval $\theta_Q \in [0,2\pi)$. We implement this by means of even reflection of the interval $\theta_Q \in [0,\pi]$ with respect to $\theta_Q = \pi$ as follows:

$$F_1\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right)=\frac{1}{2}\cdot\begin{cases} f_1\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right), & if\ \theta_Q\in[0,\pi] \\ f_1\left(2\pi-\theta_P,\varphi_P,2\pi-\theta_Q,\varphi_Q\right), & if\ \theta_Q\in(\pi,2\pi)\end{cases} \tag{9}$$

and

$$F_2\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right)=\frac{1}{2}\cdot\begin{cases} f_2\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right), & if\ \theta_Q\in[0,\pi] \\ f_2\left(2\pi-\theta_P,\varphi_P,2\pi-\theta_Q,\varphi_Q\right), & if\ \theta_Q\in(\pi,2\pi)\end{cases} \tag{10}$$

Now the functions $F_1\left(\theta_Q,\varphi_Q\right)$ and $F_2\left(\theta_Q,\varphi_Q\right)$ are defined in the domain $\left(\theta_Q,\varphi_Q\right)\in[0,2\pi)\times[0,2\pi)$ and it is possible to expand them into the Fourier series (using evenness of the functions with respect to $\theta_Q=\pi$):

$$F_1\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right)\approx c_0(P)+\sum_{n=1}^{N_1}\sum_{m=1}^{M_1}\left[c_{mn}(\theta_P,\varphi_P)\cos\left(m\varphi_Q\right)+d_{mn}(\theta_P,\varphi_P)\sin\left(m\varphi_Q\right)\right]\cdot\cos\left(n\theta_Q\right),$$
$$F_2\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right)\approx f_0(P)+\sum_{n=1}^{N_1}\sum_{m=1}^{M_1}\left[f_{mn}(\theta_P,\varphi_P)\cos\left(m\varphi_Q\right)+h_{mn}(\theta_P,\varphi_P)\sin\left(m\varphi_Q\right)\right]\cdot\cos\left(n\theta_Q\right) \tag{11}$$

where $c_0(\theta_P,\varphi_P), c_{mn}(\theta_P,\varphi_P), d_{mn}(\theta_P,\varphi_P)$ and $f_0(\theta_P,\varphi_P), f_{mn}(\theta_P,\varphi_P), h_{mn}(\theta_P,\varphi_P)$, $(m=1,2,\ldots,M_1>M;\, n=1,2,\ldots,N_1>N)$ could be obtained by a discrete Fourier transform for every particular point P. In the case $M_1=2^{m_1}, N_1=2^{n_1}$, where m_1, n_1- integers, the algorithms of fast Fourier transformations could be implemented.

Let us redefine the surface velocity $V_n(Q)$ in a similar way and expand it into the Fourier series:

$$\widetilde{V_n(Q)}=\frac{1}{2}\cdot\begin{cases} V_n\left(\theta_Q,\varphi_Q\right), & if\ \theta_Q\in[0,\pi] \\ V_n\left(2\pi-\theta_Q,\varphi_Q\right), & if\ \theta_Q\in(\pi,2\pi)\end{cases}\approx V_0+\sum_{n=1}^{N_1}\sum_{m=1}^{M_1}\left[V_{mn}^{(c)}\cos\left(m\varphi_Q\right)+V_{mn}^{(s)}\sin\left(m\varphi_Q\right)\right]\cdot\cos\left(n\theta_Q\right) \tag{12}$$

Now equation (6) could be rewritten as follows:

$$\widetilde{C(\theta_P,\varphi_P)}\cdot\Phi(\theta_P,\varphi_P)\approx\int_0^{2\pi}\int_0^{2\pi}\left[F_1\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right)\cdot\Phi\left(\theta_Q,\varphi_Q\right)+F_2\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right)\cdot\widetilde{V_n\left(\theta_Q,\varphi_Q\right)}\right]d\theta_Q\, d\varphi_Q$$
$$+V_n(\theta_P,\varphi_P)\cdot\int_{\varphi_P-\Delta\varphi}^{\varphi_P+\Delta\varphi}\int_{\theta_P-\Delta\theta}^{\theta_P+\Delta\theta}g\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right)\cdot\frac{r^2\left(\theta_Q,\varphi_Q\right)}{\cos\left[\overrightarrow{r\left(\theta_Q,\varphi_Q\right)},\overrightarrow{n_Q}\right]}\cdot\sin\theta_Q\, d\theta_Q\, d\varphi_Q+4\pi\cdot\Psi^{(i)}(\theta_P,\varphi_P) \tag{13}$$

where

$$\widetilde{C(\theta_P,\varphi_P)}=C(\theta_P,\varphi_P)-\int_{\varphi_P-\Delta\varphi}^{\varphi_P+\Delta\varphi}\int_{\theta_P-\Delta\theta}^{\theta_P+\Delta\theta}\frac{\partial\ g\left(\theta_P,\varphi_P,\theta_Q,\varphi_Q\right)}{\partial n_Q}\cdot\frac{r^2\left(\theta_Q,\varphi_Q\right)}{\cos\left[\overrightarrow{r\left(\theta_Q,\varphi_Q\right)},\overrightarrow{n_Q}\right]}\cdot\sin\theta_Q\, d\theta_Q\, d\varphi_Q \tag{14}$$

and $C(\theta_P,\varphi_P)=C(P)$ is known from the geometry of the structure (see (2)).

Explicit representation of the Green function in spherical coordinates is $g = e^{-ik\rho} / \rho$, where

$$\rho=\sqrt{r^2\left(\theta_Q,\varphi_Q\right)+r^2\left(\theta_P,\varphi_p\right)-2\cdot r\left(\theta_Q,\varphi_Q\right)\cdot r\left(\theta_P,\varphi_p\right)\cdot\left[\cos\theta_Q\cos\theta_P+\sin\theta_Q\sin\theta_P\cos\left(\varphi_Q-\varphi_p\right)\right]} \tag{15}$$

The explicit form of the normal derivative of the Green function is

$$\frac{\partial\ g}{\partial n_Q}=\frac{\partial}{\partial\rho}\left(\frac{e^{-ik\rho}}{\rho}\right)\cdot\frac{\partial\ \rho}{\partial n_Q}=-ik\cdot\left\{g\cdot\cos\left[\left(\overrightarrow{P,Q}\right),\overrightarrow{n_Q}\right]\right\}-g\cdot\left\{\frac{1}{\rho}\cdot\cos\left[\left(\overrightarrow{P,Q}\right),\overrightarrow{n_Q}\right]\right\} \tag{16}$$

Substituting (8), (11) - (12) in (13) and using orthogonality of *sin* - and *cos* – functions wee obtain the following linear algebraic equation with unknowns a_0, a_{mn} and b_{mn}:

$$p_0(P)\cdot a_0+\sum_{n=1}^{N_1}\sum_{m=1}^{M_1}\left[p_{mn}(P)\cdot a_{mn}+q_{mn}(P)\cdot b_{mn}\right]\approx r(P) \tag{17}$$

where

$$p_0(P)=\tilde{C}(P)-4\pi^2c_0;$$
$$p_{mn}(P)=\tilde{C}(P)\cos\left(m\varphi_P\right)\cos\left(n\theta_P\right)-\pi^2c_{mn};$$
$$q_{mn}(P)=\tilde{C}(P)\sin\left(m\varphi_P\right)\cos\left(n\theta_P\right)-\pi^2d_{mn};$$
$$r(P)=\pi^2\left\{4f_0(P)V_0(P)+\sum_{n=1}^{N_1}\sum_{m=1}^{M_1}\left[f_{mn}(P)\cdot V_{mn}^{(c)}(P)+h_{mn}(P)\cdot V_{mn}^{(s)}(P)\right]\right\}$$
$$+V_n(P)\cdot\int_{\varphi_P-\Delta\varphi}^{\varphi_P+\Delta\varphi}\int_{\theta_P-\Delta\theta}^{\theta_P+\Delta\theta}g(P,Q)\cdot\frac{r^2(Q)}{\cos\left[\overrightarrow{r(Q)},\overrightarrow{n_Q}\right]}\cdot\sin\theta_Q\,d\theta_Q\,d\varphi_Q+4\pi\Psi^{(i)}(P). \tag{18}$$

Changing point *P*, a new equation could be obtained, etc. It is recommended to randomize location of points *P* on the surface of the structure so to obtain $2\cdot M\cdot N+1$ or more equations and solve them by means of a least squares method.

It is necessary to stress that for all continuous surfaces normal derivative $\frac{\partial\ \rho}{\partial n_Q}\xrightarrow[P\to Q]{}0$ and products $\left(\frac{1}{\rho}\cdot\frac{\partial\ \rho}{\partial n_Q}\right)$ and $\left(g\cdot\frac{\partial\ \rho}{\partial n_Q}\right)$ are finite at $P\to Q$ but discontinuous in general case or the so-called functions of finite variation. So the singularity in $\frac{\partial\ g}{\partial n_Q}$ is situated only due to the second term (g) in (16). Let us show the example of the discontinuity of $\left(\frac{1}{\rho}\cdot\frac{\partial\ \rho}{\partial\, n_Q}\right)$. We consider a circular cylinder of radius r with axis Oz. Suppose that point P is located on cross-section of axis Ox and the cylindrical surface so that its coordinates are $[r, 0, 0]^T$. Point Q is also located on the cylindrical surface with coordinates $[r\cos\varphi, r\sin\varphi, z]^T$. Hence, $\|\vec{\rho}\| = \sqrt{4r^2\sin^2\left(\frac{\varphi}{2}\right)+z^2}$. The vector of outer unit normal at point Q is $\vec{n}_Q = [r\cos\phi, r\sin\varphi, 0]^T$ and $\frac{1}{\rho}\frac{\partial\ \rho}{\partial\, n_Q} = \frac{1}{\|\vec{\rho}\|}\frac{\partial\ \rho}{\partial\, n_Q} = \frac{2r\cdot\sin^2\left(\frac{\varphi}{2}\right)}{4r^2\cdot\sin^2\left(\frac{\varphi}{2}\right)+z^2}$. In *Figure 1* the graph of this function is shown in (r,z)- coordinates. The function is discontinuous in the point $(0,0)$.

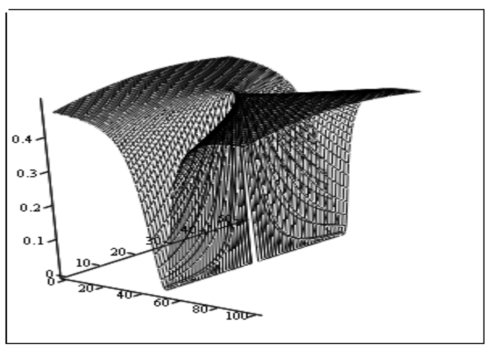

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Figure 1- Discontinuity of Normal Derivative of Green Function

EXAMPLE

Let us consider pressure distributions of a plane incident wave in the vicinity of a cylinder with hemispherical end caps (Fig 2). Axis of the structure is supposed to be horizontal. Radius of the cylinder and hemispheres $a = 3\,m$ and distance between the spherical poles is $h = 54\,m$. Incident angle equal of plane incident wave is $\alpha = 30\ deg$. Frequency of incident wave is $f = 500\,Hz$. Horizontal axis – θ- angle of the structure $(\theta \in [0,\pi])$, vertical axis – polar angle of the structure $(\varphi \in [0,2\pi])$. Left pictures show the pressure distribution in the presence of the structure. Right pictures show the pressure distribution in the acoustic field in the absence of the structure. Comparison of these pictures helps to estimate the level of the acoustic field scattering by the structure. Colour pictures are accompanied by the digital maps (black and white figures). All figures are given in the linear normalized scale.

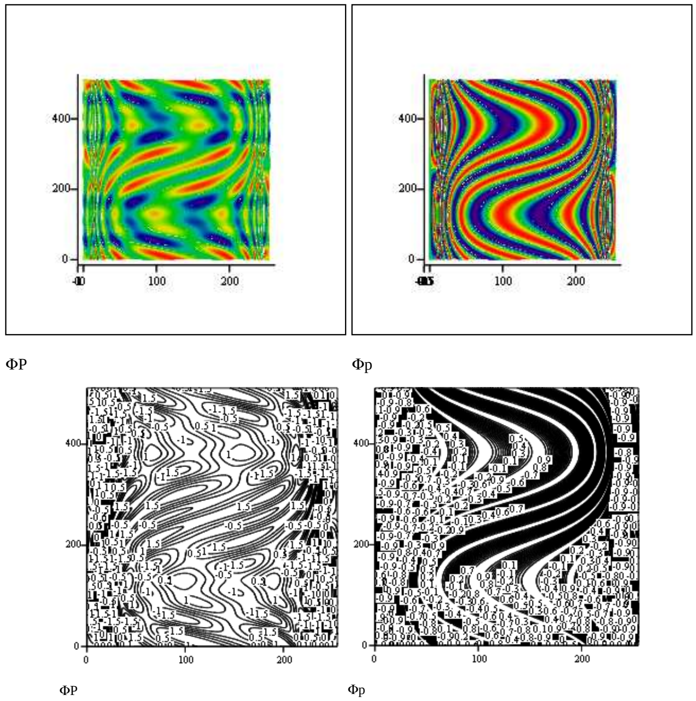

Figure 2 – Pressure distribution in the absence of scattered structure(right) and in the presence of the structure (left) and their digital maps

CONCLUSIONS

The method of solution of the Helmholtz integral equation is formulated, which is based on representation of a velocity potential in terms of Fourier series and finding the Fourier coefficients of this expansion. The Green function is modified so to satisfy the Dirichlet's theorem. Fourier coefficients of the modified Green functions are calculated using a discrete Fourier transform, in particular case by a fast Fourier transformation. Using orthogonality of the sine and cosine functions the original problem is reduced to an overdetermined system of linear algebraic equations to obtain the unknown coefficients of the Fourier series expansion. This method is applicable to a broad range of acoustical problems of radiation and scattering. It can be easily parallelized and realized on grid or vector computers. The example of calculation of near acoustic fields of large-scale structures is given.

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