Vortices in Gaussian beams

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- ▷ Definition of polynomial Gaussian beam
- ▷ Calculation techniques and general properties of polynomial Gaussian beam
- ▷ Single vortex beams
- ▷ Gaussian beams with vortex dipoles

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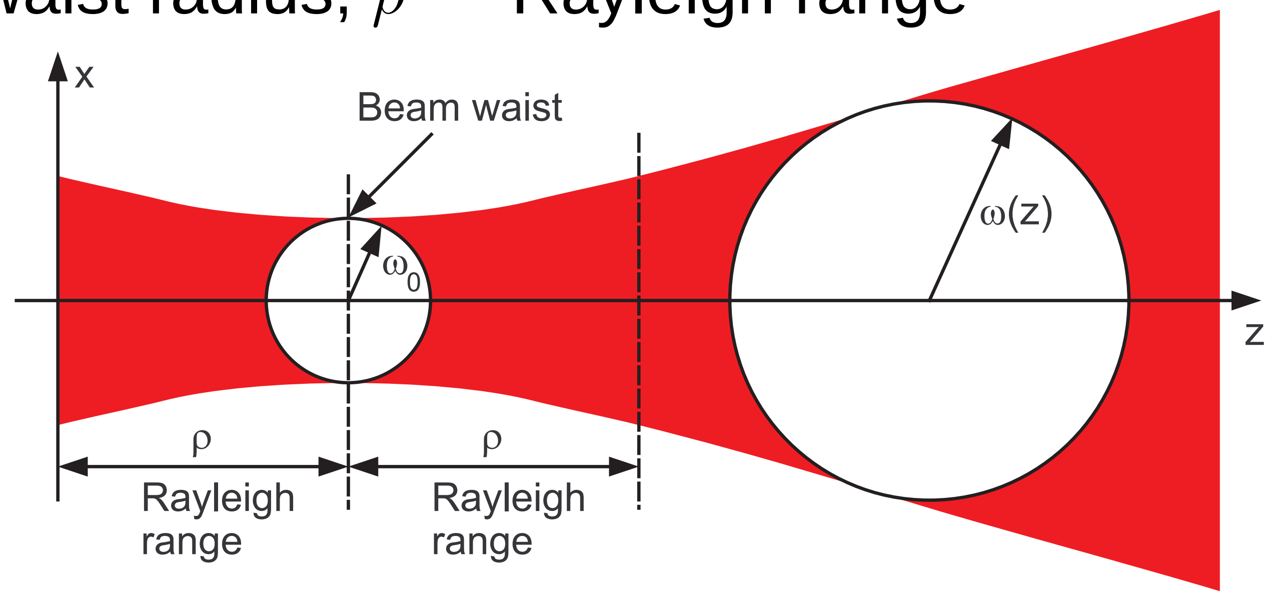
Gaussian beam notation

Gaussian beam in normalised coordinates:

$$g(u,v,t) = \exp\left(-\frac{u^2+v^2}{1-it}\right)$$

$$u = \frac{x}{\omega_0} \quad v = \frac{y}{\omega_0} \quad t = \frac{z}{\rho} \quad \rho = \frac{\pi\omega_0^2}{\lambda}$$

ω_0 — $1/e^2$ beam waist radius; ρ — Rayleigh range



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Gaussian beam

Gaussian beam in terms of amplitude and phase

$$g(u,v,t) = \exp\left(-\frac{u^2+v^2}{1+t^2}\right)\exp\left(-\frac{it(u^2+v^2)}{1+t^2}\right)$$

Normalised beam radius:

$$\sqrt{1+t^2}$$

Wavefront normalised radius of curvature:

$$R(t) = \frac{1+t^2}{t}$$

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Polynomial prefactor

Polynomial Gaussian beam = prefactor $\times$ Gaussian beam

$$f(u,v,t) = P(u,v,t)g(u,v,t)$$

$g(u,v,t)$ — Gaussian beam

$P(u,v,t)$ — complex bivariate polynomial (of finite order)

"bivariate" means two variables (u and v). Coefficients depend on t. Bivariate polynomials cannot always be factorised.

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Complex bivariate polynomials

General form of complex bivariate polynomial prefactor:

$$P(u,v,t) = \sum_{p=0}^{N} \sum_{q=0}^{p} \alpha_{p,q}(t)(u+iv)^{p-q}(u-iv)^{q}$$

In terms of $(u+iv)$ and $(u-iv)$ instead of u and v

$\alpha_{p,q}(t)$ — complex coefficients as functions of the propagation distance t

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Helical coordinates

Occasionally it will be useful to re-express polynomial Gaussian beam in terms of helical coordinates:

$$f(w,\overline{w},t) = P(w,\overline{w},t)\exp\left(-\frac{w\overline{w}}{1-it}\right)$$

where $w = u + iv$ and $\overline{w} = u - iv$; and

$$P(w,\overline{w},t) = \sum_{p=0}^{N}\sum_{q=0}^{p}\alpha_{p,q}(t)w^{p-q}\overline{w}^{q}$$

Complex bivariate polynomials

$$
\begin{aligned}
p_4(w,\overline{w}) \quad = \quad & \alpha_{00} \\
& + \alpha_{10} w + \alpha_{11} \overline{w} \\
& + \alpha_{20} w^2 + \alpha_{21} w \overline{w} + \alpha_{22} \overline{w}^2 \\
& + \alpha_{30} w^3 + \alpha_{31} w^2 \overline{w} + \alpha_{32} w \overline{w}^2 + \alpha_{33} \overline{w}^3 \\
& + \alpha_{40} w^4 + \alpha_{41} w^3 \overline{w} + \alpha_{42} w^2 \overline{w}^2 + \alpha_{43} w \overline{w}^3 + \alpha_{44} \overline{w}^4
\end{aligned}
$$

Leading order by itself is fully factorisable in terms of product of 1st order polynomials. Each 1st order polynomial represents a complex zero $\Rightarrow$ an optical vortex.

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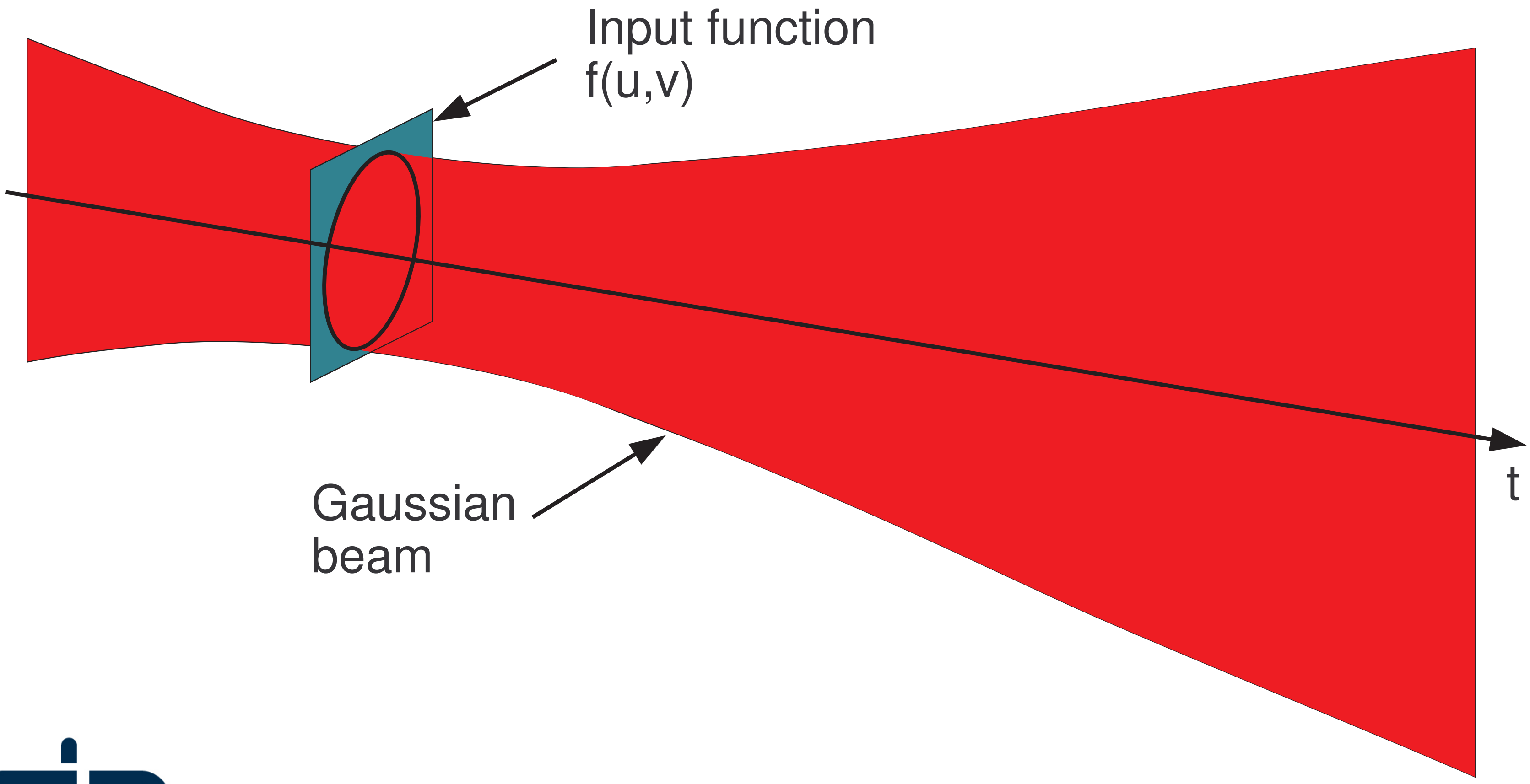
Propagation

We want to find out what the t-dependence is for a beam with a certain input function at $t = t_0$.



Fresnel integral

$$g(u',v',t) = \Omega Q(u',v',t) \iint_{-\infty}^{\infty} f(u,v) Q(u,v,t) K(u,v,t)\ du dv$$

Ω — t-dependent complex factor
$f(u,v)$ — two-dimensional complex input function at $t = t_0$

$$Q(u,v,t) = \exp\left[-\frac{i}{t-t_0}\left(u^2+v^2\right)\right]$$

$$K(u,v,t) = \exp\left[\frac{i2}{t-t_0}\left(uu'+vv'\right)\right]$$

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Propagating Gaussian beams

Fresnel propagation of polynomial Gaussian beam is represented by,

$$g(u', v', t) = \Omega Q(u', v', t)\mathcal{F}\mathcal{R}\left\{P(u, v)g(u, v, t_0)\right\}$$

$P(u, v)$ — polynomial prefactor in the input plane (at $t = t_0$)

$g(u, v, t_0)$ — Gaussian beam in the input plane (at $t = t_0$)

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Replacement

Replace variable with derivative

$$u \exp\left(\frac{i2}{t-t_0}uu'\right) = \frac{t-t_0}{i2}\frac{\partial}{\partial u'}\exp\left(\frac{i2}{t-t_0}uu'\right)$$

Since the derivatives are independent of the integration variables they can be pulled out of the integral.

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Operator approach

One can simplify calculations, by turning polynomial prefactor into differential operator:

$$\mathcal{FR}\left\{P(u,v)g(u,v,t_0)\right\} = \mathbf{P}(d_u,d_v)\left\{\mathcal{FR}\left\{g(u,v,t_0)\right\}\right\}$$

Replacement:

$$u \rightarrow d_u = \frac{t-t_0}{i2}\frac{\partial}{\partial u'}$$

$$v \rightarrow d_v = \frac{t-t_0}{i2}\frac{\partial}{\partial v'}$$

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Differentiation i.s.o integration

Evaluate the integral over the Gaussian beam (once and for all). Then, instead of evaluating an integral every time, one only needs to perform differentiations:

$$g(u,v,t) = \Omega Q(u,v,t)\left(\frac{t_0+i}{t+i}\right)\mathbf{P}\left(d_u,d_v\right)\{\Gamma(u,v,t)\}$$

where we replace (u',v') with (u,v).

$$\Gamma(u,v,t) = \exp\left[i\frac{(t_0+i)(u^2+v^2)}{(t+i)(t-t0)}\right]$$

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Helical coordinates

For propagation:

$$g(w,\overline{w},t)=\Omega\exp\left(-\frac{iw\overline{w}}{t-t_0}\right)\left(\frac{t_0+i}{t+i}\right)\mathbf{P}\left(d_w,d_{\overline{w}}\right)\left\{\Gamma(w,\overline{w},t)\right\}$$

where

$$\Gamma(w,\overline{w},t)=\exp\left[i\frac{(t_0+i)w\overline{w}}{(t+i)(t-t0)}\right]$$

$$w'\rightarrow d_w=-i(t-t_0)\frac{\partial}{\partial\overline{w}}$$

$$\overline{w}'\rightarrow d_{\overline{w}}=-i(t-t_0)\frac{\partial}{\partial w}$$

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Single noncanonical vortex

Prefactor at $t = t_0$:

$$p_1(w, \overline{w}) = \xi(w - w_0) + \zeta(\overline{w} - \overline{w}_0)$$

where $w_0 = u_0 + iv_0$ and $\overline{w}_0 = u_0 - iv_0$.

After propagation:

$$p_1(w, \overline{w}, t) = \xi \left(w \left[\frac{t_0 + i}{t + i} \right] - w_0 \right) + \zeta \left(\overline{w} \left[\frac{t_0 + i}{t + i} \right] - \overline{w}_0 \right)$$

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Trajectories

Steps to find the vortex trajectories:

- ▷ Substitute: w, $\overline{w}$, ξ and ζ in terms of real and imaginary parts.
- ▷ Separate prefactor into real and imaginery parts.
- ▷ Find coordinates (u, v) as function of t where both parts become zero.

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Trajectories for example

Trajectories for $p_1 = w - w_0$ (single off-axis canonical vortex).

$$p_1 = (u + iv)\left[\frac{t_0 + i}{t + i}\right] - (u_0 + iv_0)$$

Trajectory:

$$u(t) = \frac{u_0 - v_0 t_0}{1 + t_0^2} + \frac{u_0 t_0 + v_0}{1 + t_0^2} t$$

$$v(t) = \frac{u_0 t_0 + v_0}{1 + t_0^2} - \frac{u_0 - v_0 t_0}{1 + t_0^2} t$$

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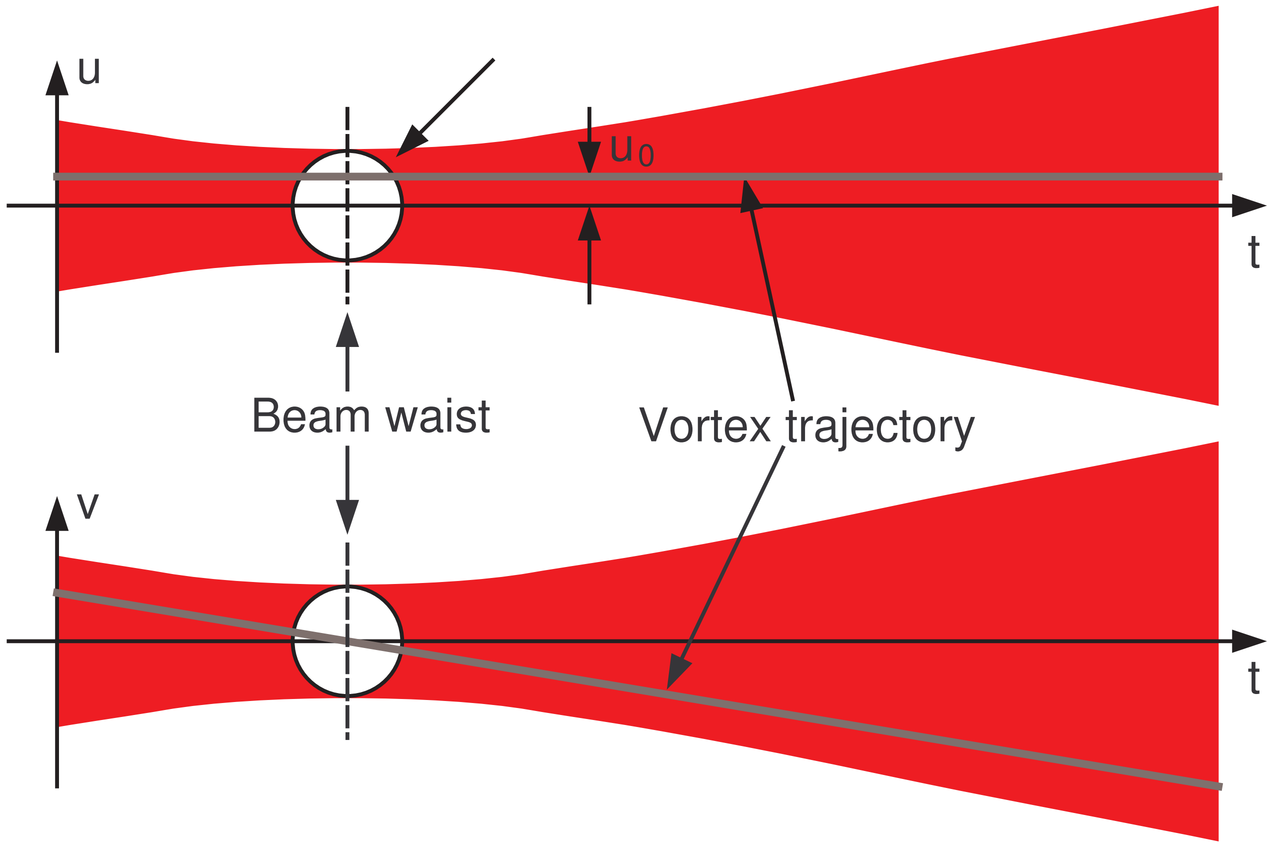
Visual example

For $v_0 = 0$ (rotate beam) and $t_0 = 0$ (input in waist):
$u(t) = u_0$ and $v(t) = -u_0 t$
The vortex propagate on a straight line lying in a plane parallel to axis.



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Vortex dipole — input

Input prefactor for vortex dipole (pair of oppositely charged vortices):

$$p_2(w,\overline{w}) = [\xi_1(w-w_1)+\zeta_1(\overline{w}-\overline{w}_1)]\,[\xi_2(w-w_2)+\zeta_2(\overline{w}-\overline{w}_2)]$$

w_1, $\overline{w}_1$, w_2, $\overline{w}_2$ — locations of the two respective vortices

ξ_1, ζ_1, ξ_2, ζ_2 — morphologies of the two respective vortices

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Vortex dipole — propagated

Propagation gives:

$$
\begin{aligned}
p_2(w,\overline{w},t) &= \left[\xi_1\left(w\left[\frac{t_0+i}{t+i}\right]-w_1\right)+\zeta_1\left(\overline{w}\left[\frac{t_0+i}{t+i}\right]-\overline{w}_1\right)\right] \\
&\quad\times\left[\xi_2\left(w\left[\frac{t_0+i}{t+i}\right]-w_2\right)+\zeta_2\left(\overline{w}\left[\frac{t_0+i}{t+i}\right]-\overline{w}_2\right)\right] \\
&\quad-(\zeta_2\xi_1+\xi_2\zeta_1)(t-t_0)\left(\frac{t_0+i}{t+i}\right)
\end{aligned}
$$

Result contains product of vortices, plus additional coupling term. Coupling term destroys factorisability

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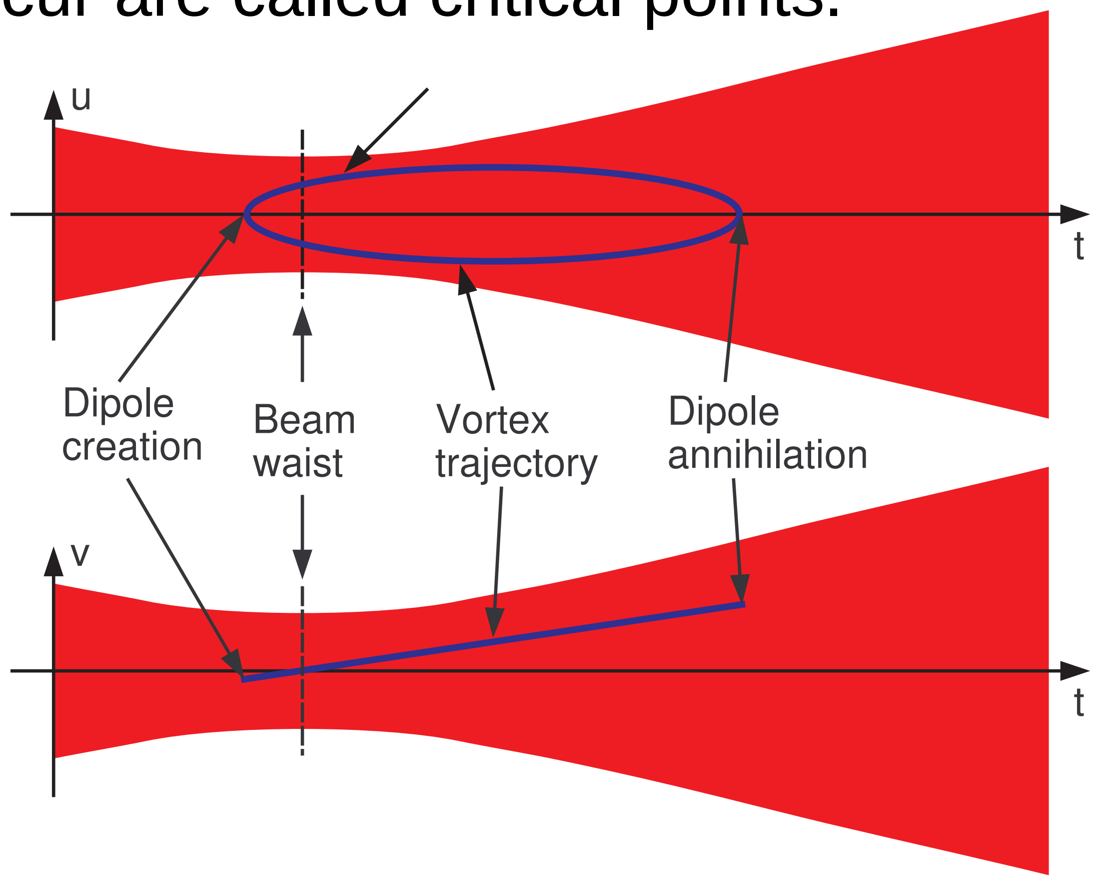
Annihilations

Coupling term $\Rightarrow$ dipole annihilations and dipole creations. Without coupling term, prefactor is always factorisable $\Rightarrow$ fixed number of vortices. Points where dipole annihilations and dipole creations occur are called critical points.



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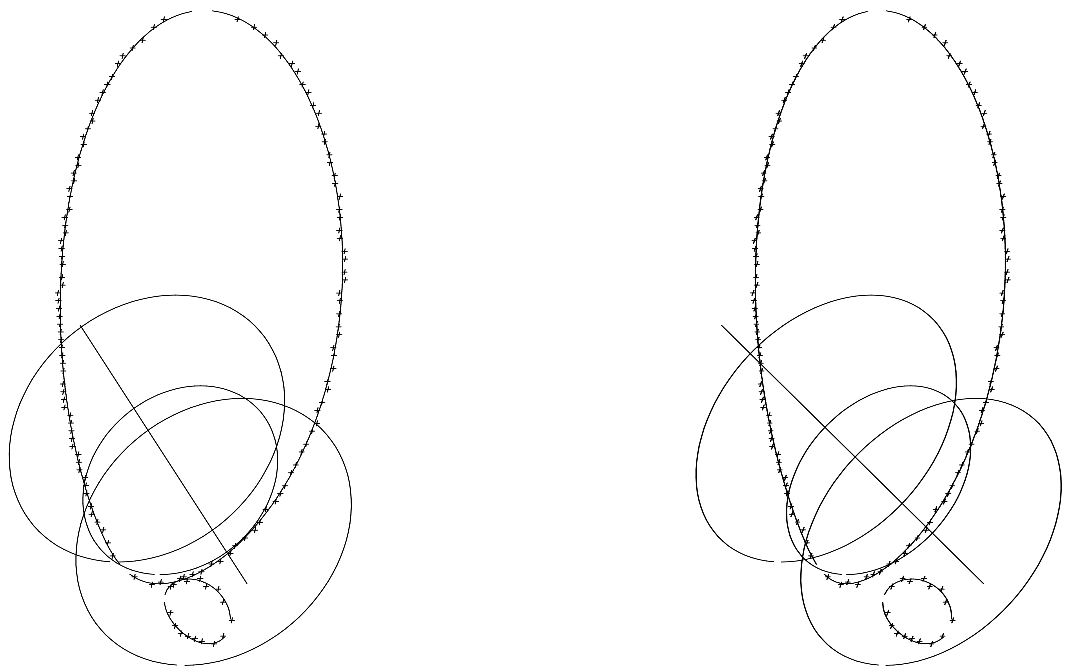
Simulated example



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Dipole propagation example

Follow same steps as before. Trajectory equations are in general second order in u and $v \Rightarrow$ only solutions when discriminant > 0

Example: two canonical vortices: $p_2 = (w - u_0)(\overline{w} + u_0)$

After propagation:

$$\begin{aligned} p_2(w, \overline{w}, t) &= \left[(u + iv)\left(\frac{t_0 + i}{t + i}\right) - u0\right] \\ &\times \left[(u - iv)\left(\frac{t_0 + i}{t + i}\right) + u0\right] \\ &-i(t - t_0)\left(\frac{t_0 + i}{t + i}\right) \end{aligned}$$

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Dipole trajectories

Trajectory:

$$v(t) = \left[\frac{1}{2u_0} - \frac{u_0}{1+t_0^2}\right](t - t_0)$$

$$u(t) = \pm\frac{\sqrt{4\left(1+t_0^2\right)\left(1+t^2\right)u_0^4 - (t-t_0)^2\left(1+t_0^2-2u_0^2\right)^2}}{2\left(1+t_0^2\right)u_0}$$

Part under square root must be positive.

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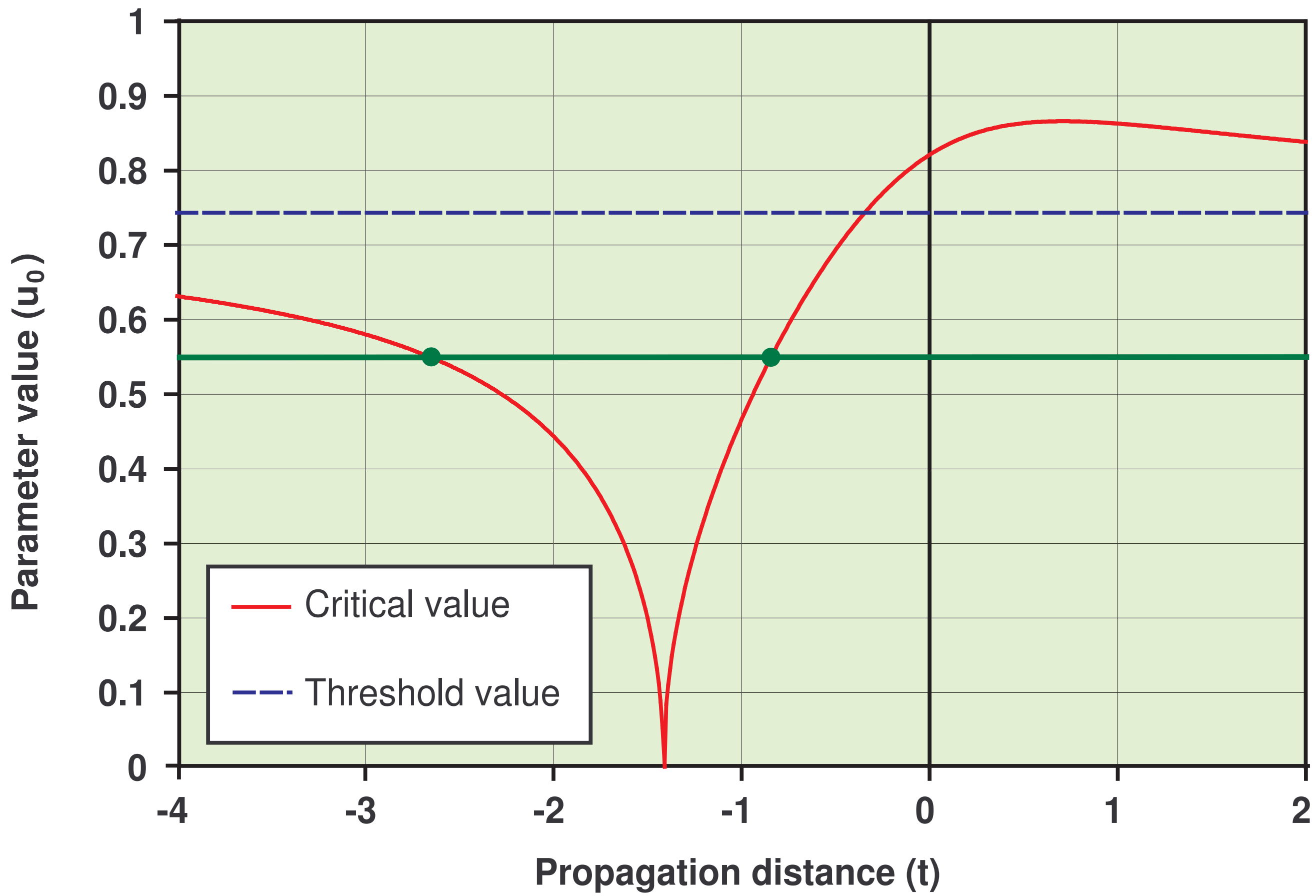
Critical point analysis

Critical points are where discriminant$= 0$. Find critical point by setting discriminant equal to 0 and solve for parameters u_0.



Morphology evolusion

The morphology for a vortex dipole changes during propagation.

The compute the morphology:

- ▷ Compute trajectories for vortex dipole
- ▷ Compute morphology distributions
- ▷ Evaluate distributions at locations on trajectory

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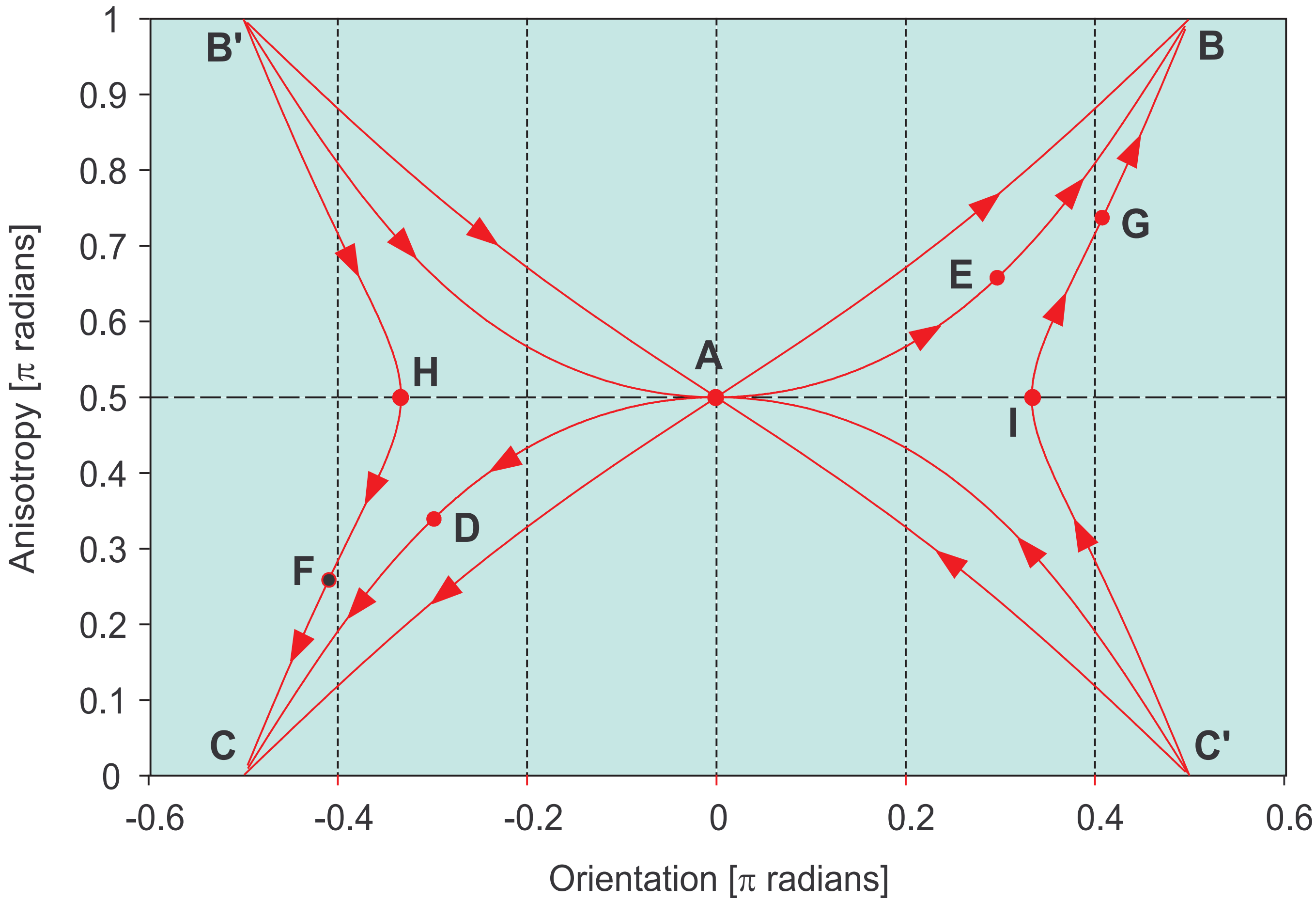
Morphology evolusion example



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Observation

- Vortices become edge dislocations (degenerate) near critical points.
- Vortices are only isotropic (canonical) in input plane.
- Morphology at $t = \infty$ is the same as the morphology at $t = -\infty$.

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Conclusions

- Polynomial Gausian beams provide a relatively easy way to investigate vortex behaviour.
- Single vortices propagate in straight lines.
- Vortex dipoles give dipole creation and dipole annihilation.
- Vortex morphologies evolve for vortex dipoles.

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