Deconstructing quantum decoherence in atmospheric turbulence

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our future through science

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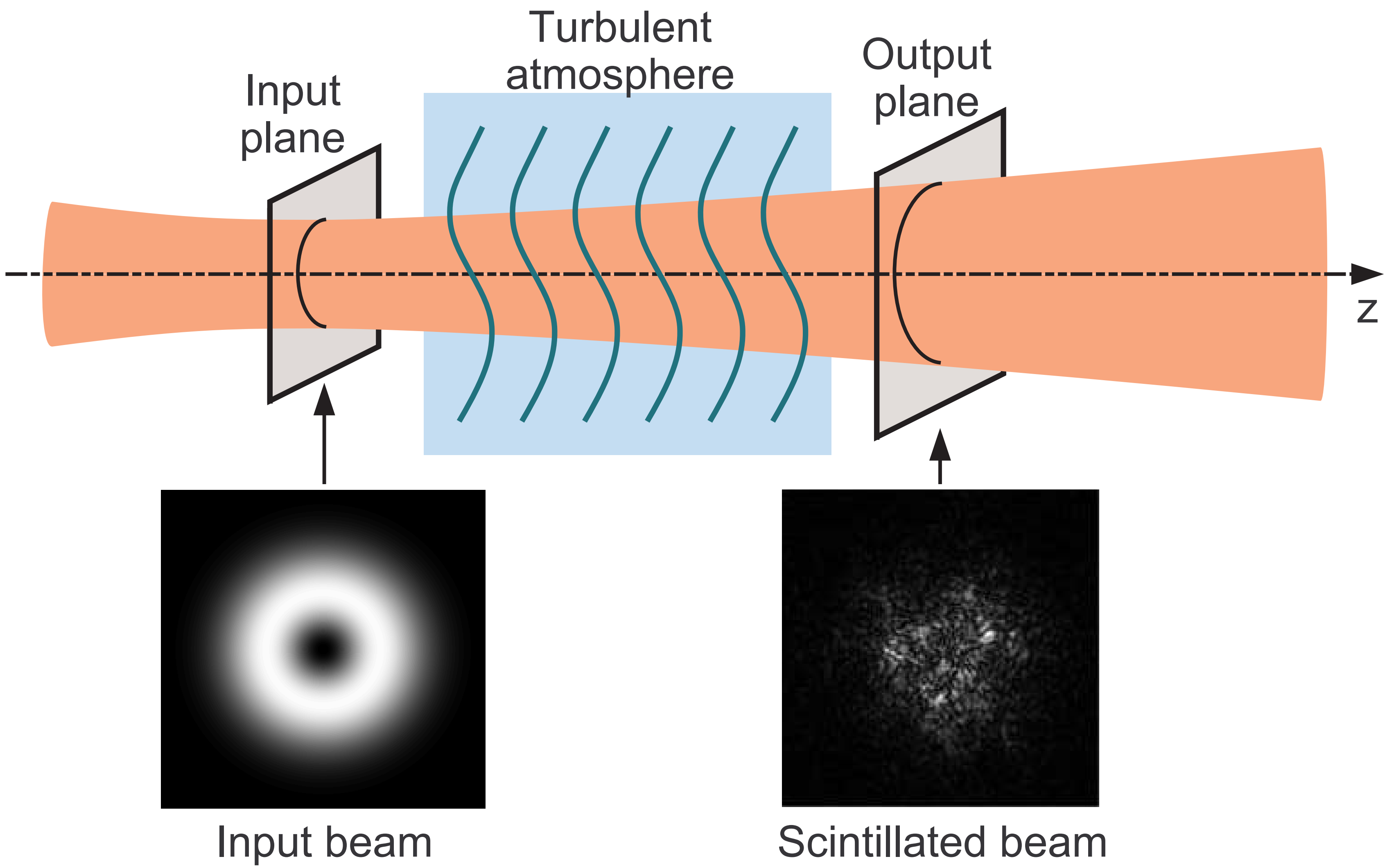
- ▷ Classical and quantum scintillation
- ▷ Time vs propagation distance
- ▷ Current paradigm: Paterson model
- ▷ Infinitesimal approach: IPE and its properties
- ▷ Optimal quantum states

Classical scintillation



Turbulence

Scintillation is caused by random phase modulations due to refractive index fluctuations in a turbulence atmosphere

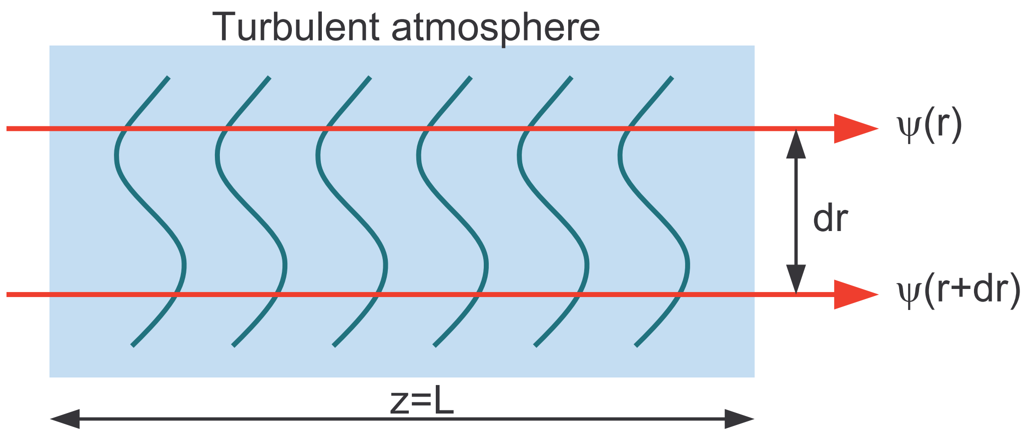
Refractive index fluctuations: $n(\mathbf{r}) = 1 + \tilde{n}(\mathbf{r})$

Any scintillation model requires a turbulence model such as, Kolmogorov, von Karman, Tartarskii[a]

Turbulence models are defined by the structure function $D(\mathbf{r})$ or the power spectral density $\Phi(\mathbf{k})$ of the refractive index fluctuations

[a] LC Andrews and RL Phillips, *Laser beam propagation through random media*, 2nd ed. SPIE Press (2005)

Turbulence model



Structure function:

$$D(\Delta\mathbf{r}) = \langle [\tilde{n}(\mathbf{r}_1) - \tilde{n}(\mathbf{r}_2)]^2 \rangle = 2B(0) - 2B(\Delta\mathbf{r})$$

where $B(\Delta\mathbf{r}) = \langle \tilde{n}(\mathbf{r}_1)\tilde{n}(\mathbf{r}_2) \rangle$ and $\Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

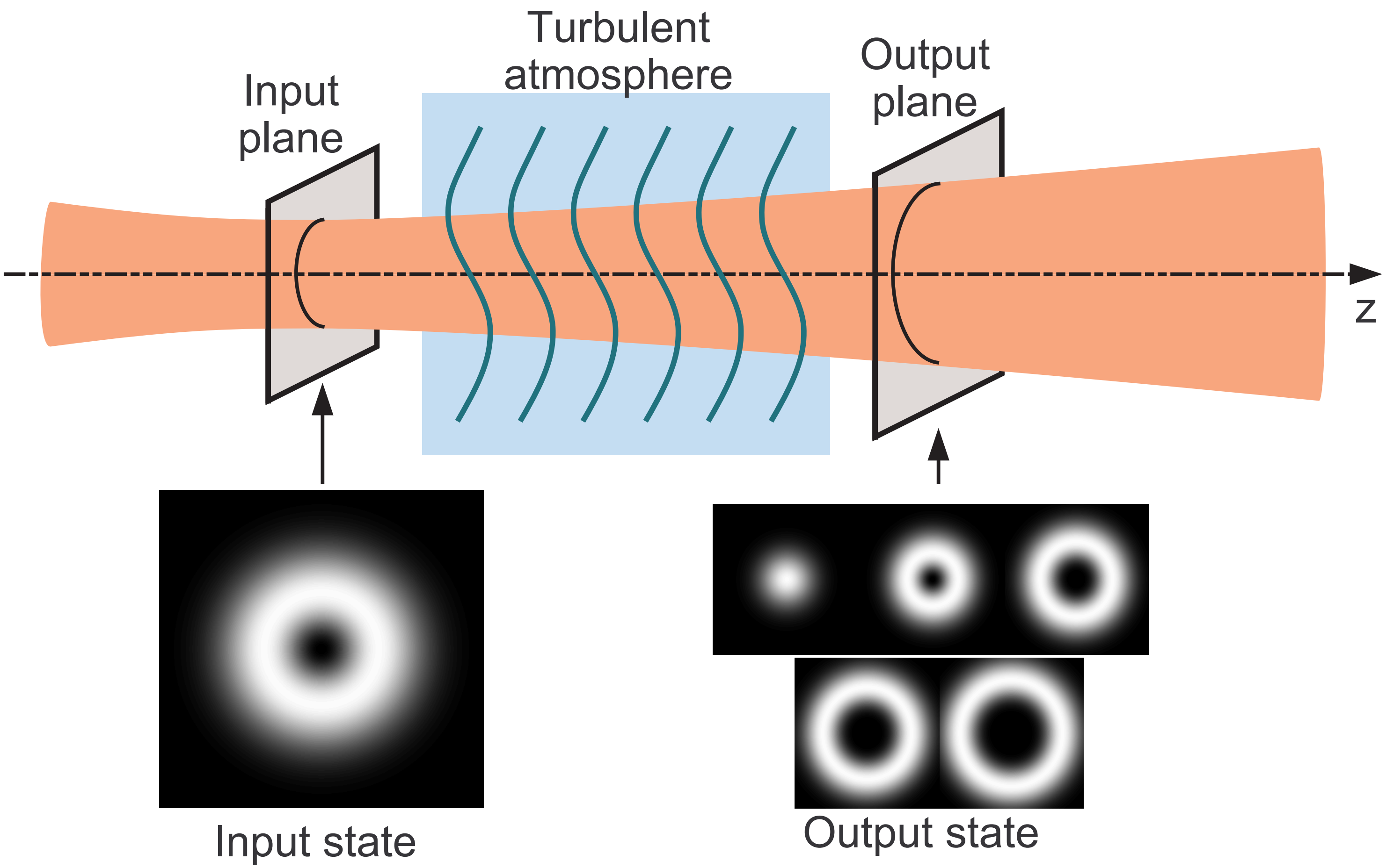
Wiener-Kintchine: $\Phi_n(\mathbf{k}) = \frac{1}{(2\pi)^3} \int B(\mathbf{r}) \exp(-i\mathbf{k}\cdot\mathbf{r})\ \mathrm{d}^3 r$

$\Phi_n(\mathbf{k})$ — Power spectral density

Kolmogorov: $\Phi_n(\mathbf{k}) = 0.033 C_n^2 |\mathbf{k}|^{-11/3}$

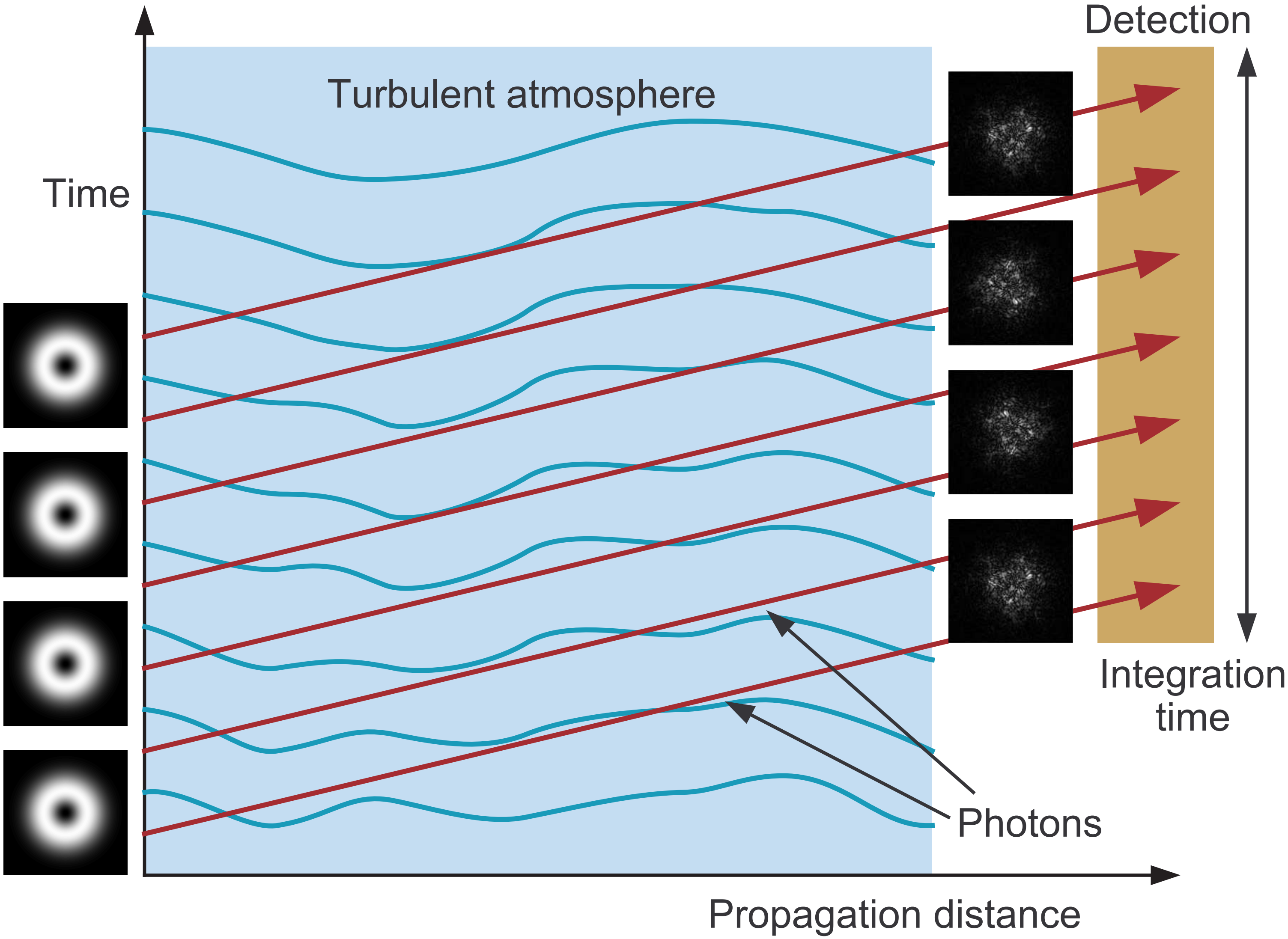
C_n^2 — Refractive index structure constant

Quantum mechanical scintillation



$$|\Psi_{\text{In}}\rangle = |l=1\rangle \qquad |\Psi_{\text{Out}}\rangle = \sum_l a_l |l\rangle$$

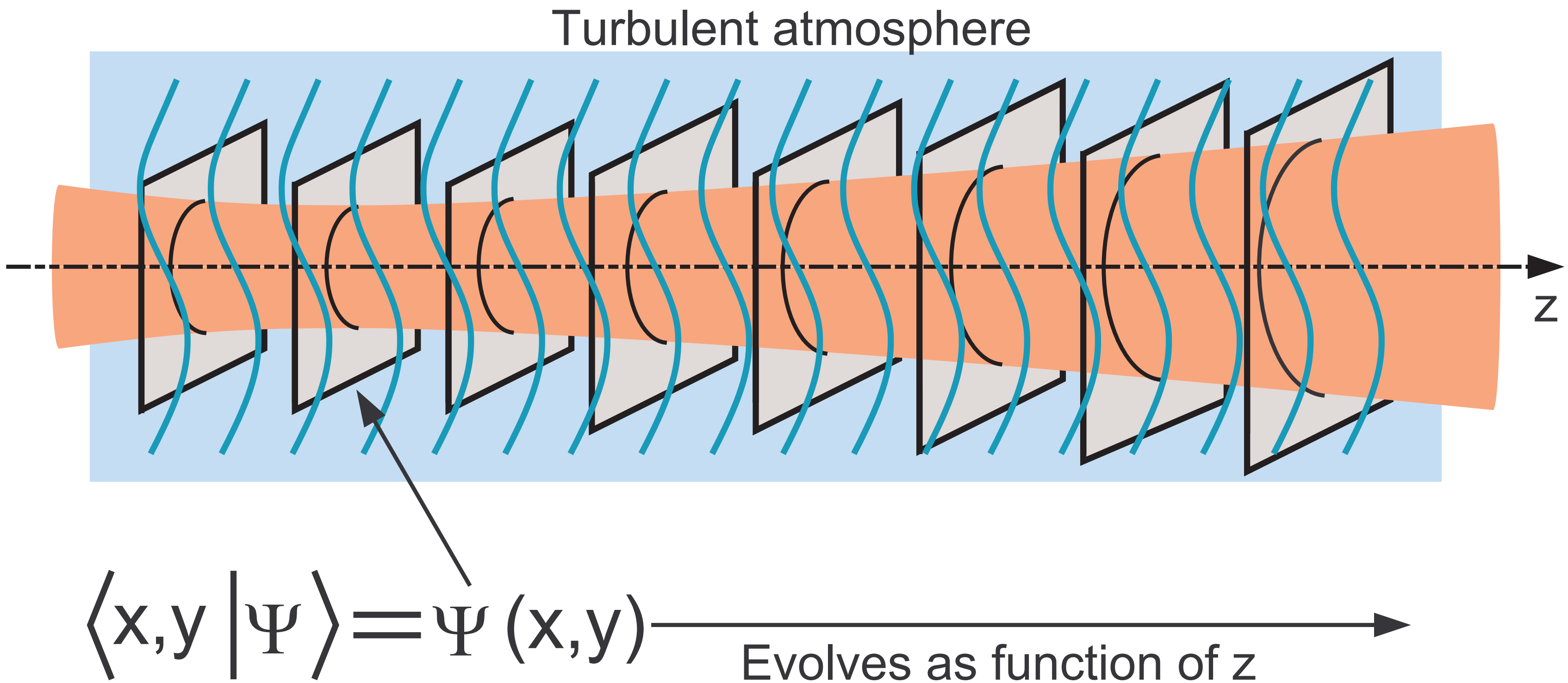
Temporal behaviour



Ensemble averaging over different instances of the medium

Wave function: Constant over time — varying along z

Evolve in space



State defined on 2D plane — evolves as function of z

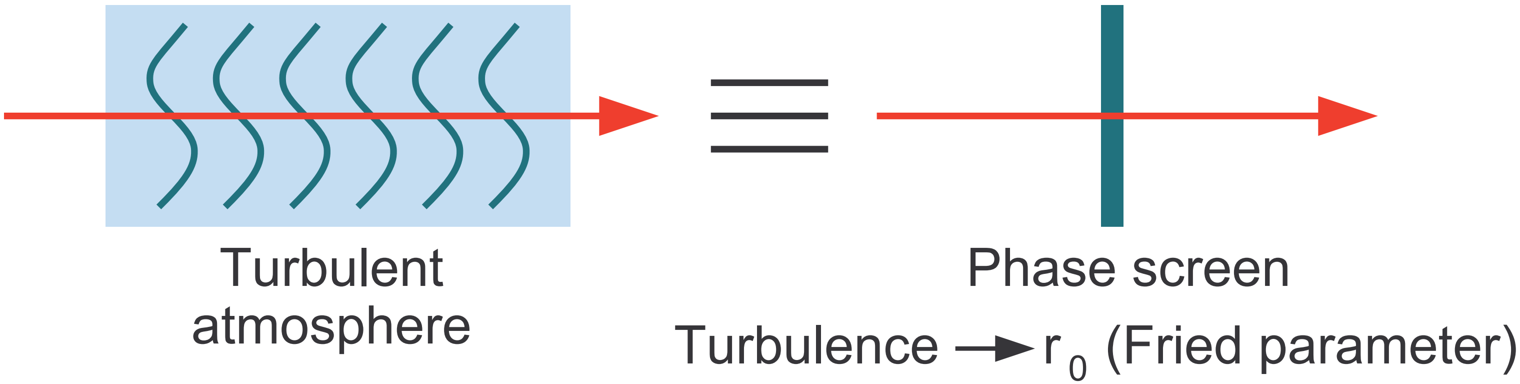
Instead of $$i\hbar\, \partial_t \rho(t) = [H, \rho(t)]$$

we need $$\partial_z \rho(z) = i\mathcal{P}\left\{\rho(z)\right\}$$

Paterson model (PM)

Assuming weak scintillation (only affects the phase)[a]

Use single phase screen:



Fried parameter (distance): $r_0 = 0.185\left(\frac{\lambda^2}{C_n^2 z}\right)^{3/5}$

Quantum operation: $\rho_{out}(z) = U\rho_{in}U^\dagger$ where $\rho_{in} = |\psi\rangle\langle\psi|$

[a]C. Paterson, Phys. Rev. Lett., **94**, 153901 (2005)

Density matrix in PM

Density matrix elements: $\rho_{mn}(z) = \langle m|U|\psi\rangle\langle\psi|U^{\dagger}|n\rangle$

$$\langle m|U|\psi\rangle = \int \langle m|\mathbf{r}\rangle\langle\mathbf{r}|U|\psi\rangle\ \mathrm{d}^2 r$$

where $\langle\mathbf{r}|m\rangle = E_m(\mathbf{r})$ with $E_m(\mathbf{r})$ — mode function

Single phase screen approximation: $\langle\mathbf{r}|U|\psi\rangle = \exp[i\theta(\mathbf{r})]\psi(\mathbf{r})$

with $\psi(\mathbf{r})$ — input beam; $\theta(\mathbf{r})$ — random phase of medium

$$\langle m|U|\psi\rangle = \int E_m^*(\mathbf{r}) \exp\left[i\theta(\mathbf{r})\right] \psi(\mathbf{r})\ \mathrm{d}^2 r$$

Density matrix element:

$$\begin{aligned}\rho_{mn}(z) \ &= \ \int\int E_m^*(\mathbf{r}_1)E_n(\mathbf{r}_2)\psi(\mathbf{r}_1)\psi^*(\mathbf{r}_2) \\ &\quad \times \exp\left[i\theta(\mathbf{r}_1) - i\theta(\mathbf{r}_2)\right]\ \mathrm{d}^2 r_1\ \mathrm{d}^2 r_2\end{aligned}$$

Ensemble averaging in PM

After ensemble averaging

$$\langle \exp\left[i\theta(\mathbf{r}_1) - i\theta(\mathbf{r}_2)\right]\rangle = \exp\left[-\frac{1}{2}D\left(|\mathbf{r}_1 - \mathbf{r}_2|\right)\right]$$

Structure function: $$D(x) = 6.88\left(\frac{x}{r_0}\right)^{5/3}$$

Ensemble averaged density matrix element:

$$\begin{aligned} \rho_{mn}(z) &= \int\int E_m^*(\mathbf{r}_1)E_n(\mathbf{r}_2)\psi(\mathbf{r}_1)\psi^*(\mathbf{r}_2) \\ &\quad \times \exp\left[-\frac{1}{2}D\left(|\mathbf{r}_1 - \mathbf{r}_2|\right)\right] \, \mathrm{d}^2 r_1 \, \mathrm{d}^2 r_2 \end{aligned}$$

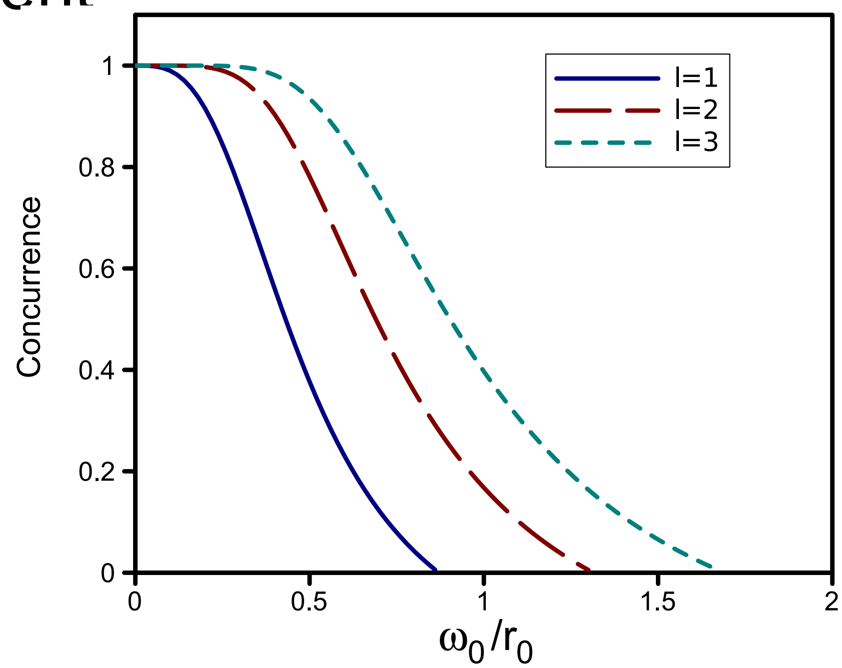
Deconstructing PM

Some observations about the Paterson model:

- Only depends on ω_0/r_0
  No other adjustable dimension parameters
  (Follows by defining dimensionless integration variables)
- Full z-dependence inside ω_0/r_0, which is inside $D(\cdot)$
  Modes are evaluated at $z = 0$
- Ensemble averaging connects U with $U^\dagger$ into one tensor
- $\Rightarrow$ Concurrence decays as function of ω_0/r_0 only
  Decays to zero (sudden death) at $\omega_0/r_0 \approx 1$
  Last longer for larger azimuthal indices

Concurrence decay in PM

Decay of qubit OAM entanglement (concurrence $\mathcal{C}$) in turbulence[a]



Quadratic structure function approximation:

$$D \sim \left(\frac{x}{r_0}\right)^{5/3} \to \left(\frac{x}{r_0}\right)^{2}$$

For point where $\mathcal{C} \to 0$:

- ▷ If C_n^2 is small $\Rightarrow$ distance z is large
- ▷ If distance z small $\Rightarrow$ C_n^2 is large

Is the approximation still valid where $\mathcal{C} \to 0$?

[a] B.J. Smith and M.G. Raymer, Phys. Rev. A, **74**, 062104 (2006)

Rytov variance

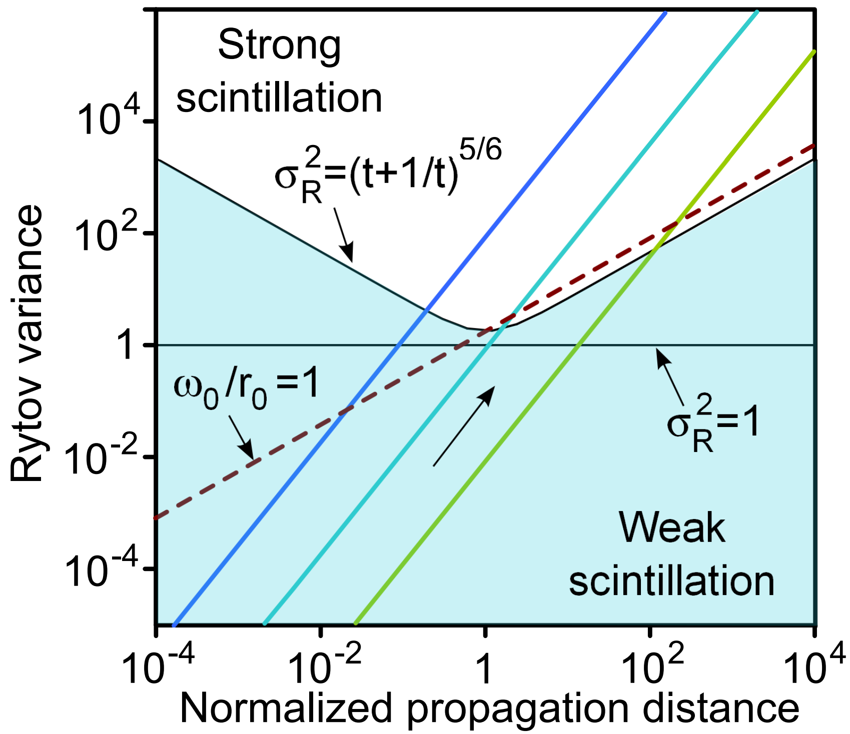
To distinguish between strong and weak scintillation for Gaussian modes with radius ω_0, one can use the Rytov variance:

$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} z^{11/6} = 1.637\, t^{5/6} \left(\frac{\omega_0}{r_0}\right)^{5/3}$$

where t is the normalized propagation distance: $t = \frac{\lambda z}{\pi \omega_0^2}$

Weak scintillation:
$\sigma_R^2 < (t + 1/t)^{5/6}$ (and $\sigma_R^2 < 1$?)
Strong scintillation:
$\sigma_R^2 > (t + 1/t)^{5/6}$ (or $\sigma_R^2 > 1$?)

Concurrence decays at about
$\sigma_R^2 t^{-5/6} = 1.637(\omega_0/r_0)^{5/3} \approx 1$



Is PM good enough?

If concurrence always decays before scintillation becomes strong shouldn't one just stay with the Paterson model?

- ▷ Does the weak/strong boundary apply to quantum entanglement?
  $\rightarrow$ numerical simulations
- ▷ What about higher azimuthal indices (large OAM)?
- ▷ How big is the error due to the quadratic structure function approximation?
  $\rightarrow$ infinitesimal approach

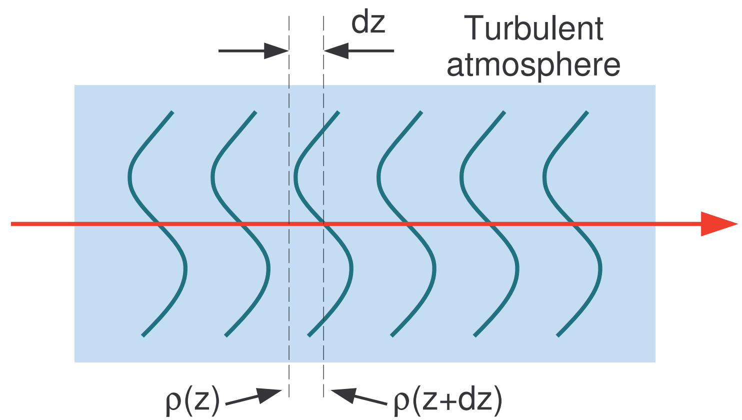
Infinitesimal approach

Consider again the single phase screen integral

$$\rho_{mn}(z) = \int E_m^*(\mathbf{r}_1)E_n(\mathbf{r}_2)\psi(\mathbf{r}_1)\psi^*(\mathbf{r}_2)\exp\left(-D/2\right)\ \mathrm{d}^2r_1\ \mathrm{d}^2r_2$$

where $D = 144\ zC_n^2|\mathbf{r}_1 - \mathbf{r}_2|^{5/3}/\lambda^2 = zD_0(\mathbf{r}_1 - \mathbf{r}_2)$

Now, instead of going from 0 to z in 1 step, we proceed in many small steps of dz



$$\rho_{mn}(z_0 + dz) = \int E_m^*(\mathbf{r}_1)E_n(\mathbf{r}_2)\sum_{pq}\rho_{pq}(z_0)E_p(\mathbf{r}_1)E_q^*(\mathbf{r}_2)$$

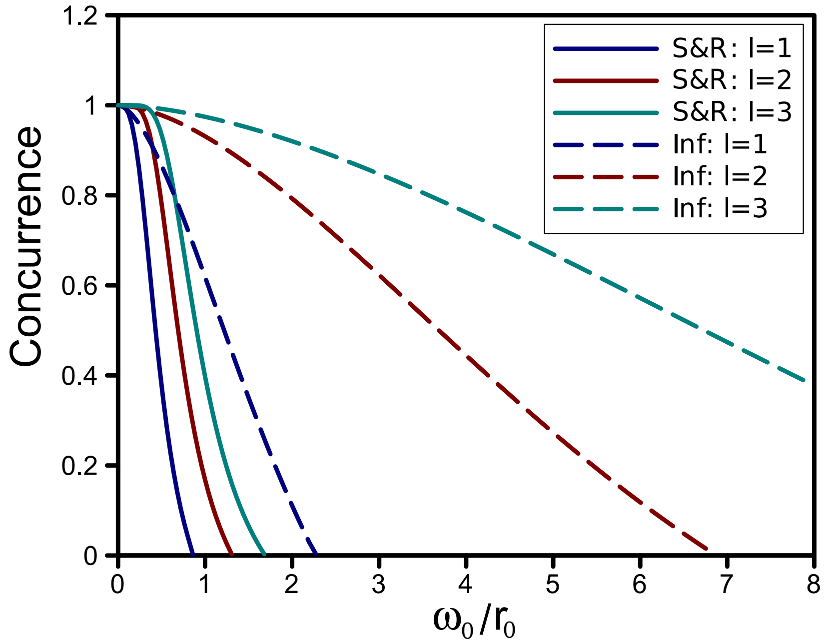
$$\times\left[1 - \frac{dzD_0(\mathbf{r}_1 - \mathbf{r}_2)}{2}\right]\ \mathrm{d}^2r_1\ \mathrm{d}^2r_2$$

Evolution equation

$$\partial_z \rho_{mn}(z) = -\frac{1}{2}\sum_{pq}\rho_{pq}(z)\int E_m^*(\mathbf{r}_1)E_n(\mathbf{r}_2) \times E_p(\mathbf{r}_1)E_q^*(\mathbf{r}_2)D_0(\mathbf{r}_1-\mathbf{r}_2)\ \mathrm{d}^2r_1\ \mathrm{d}^2r_2$$

$$= \sum_{pq}\mathcal{T}_{mnpq}\ \rho_{pq}(z)$$

1. Extend to bi-partite case
2. Solve the equation to find ρ
3. Calculate the concurrence
4. (Ignoring modal z-dependence)



$\rightarrow$ Principle behind the derivation of the IPE

The IPE

Infinitesimal propagator equation (IPE):[a]

$$\begin{aligned}\partial_z \rho_{mnpq} &= S_{xm}\rho_{xnpq} - S_{nx}\rho_{mxpq} + S_{xp}\rho_{mnxq} - S_{qx}\rho_{mnpx} \\ &\quad + L_{xymn}\rho_{xypq} + L_{xypq}\rho_{mnxy} - 2L_T\rho_{mnpq}\end{aligned}$$

First row: free-space propagation terms
Second row: dissipative terms

$$L_{mnpq} = k^2 \int \Phi_1(\mathbf{K}) W_{mp}^*(\mathbf{K}) W_{nq}(\mathbf{K}) \, \frac{\mathrm{d}^2 K}{4\pi^2}$$

$$W_{mn}(\mathbf{K}) = \int G_m(\mathbf{K}') G_n^*(\mathbf{K}' - \mathbf{K}) \, \frac{\mathrm{d}^2 K'}{4\pi^2}$$

$$\Phi_1(\mathbf{K}) = (2\pi)^3 \Phi_n(\mathbf{k}) = 0.033 (2\pi)^3 C_n^2 |\mathbf{k}|^{-11/3}$$

[a] FS Roux, Phys. Rev. A, **83**, 053822 (2011)

Properties of the IPE

- ▷ Derived in Fourier domain
  Based on power spectral density: $\Phi_n(\mathbf{k})$
- ▷ Transverse spatial modes
  $\rightarrow$ infinite dimensional Hilbert space
  $\Rightarrow$ IPE is an infinite set of coupled differential equations

  To solve them one needs to truncate the set
  $\Rightarrow$ truncated IPE is <u>not trace preserving</u>: $\mathrm{tr}\{\rho\} \leq 1$
  Some energy is scattered into excluded higher order modes
- ▷ The resulting density matrix is <u>hermitian</u>
  Follows from identity: $L_{mnpq} = L^*_{nmqp}$
- ▷ <u>Positivity</u> $\rightarrow$ only a skeleton argument yet ...

Positivity of the IPE

Infinitesimal propagation as a quantum operation:

$\rho(z+dz) = dU\ \rho(z)\ dU^\dagger$ where $dU = U(z \to z+dz)$

Ensemble averaging: $$\rho(z+dz) = \sum_n^N \frac{1}{N}\, dU_n\ \rho(z)\ dU_n^\dagger$$

where dU_n — infinitesimal propagation through different instances of medium

Since $dU_n \sim \exp(i\theta_n)$ $\Rightarrow$ $(1/N)\sum_n^N dU_n dU_n^\dagger = 1$

This has the form of an operator product expansion, which obeys positivity

$\rightarrow$ Lindblad form?

$$\partial_z\rho = i[P,\rho] + \sum_n \gamma_n (2L_n\rho L_n^\dagger - \rho L_n^\dagger L_n - L_n^\dagger L_n\rho)$$

Deconstructing the IPE

Density matrix elements (one photon state):

$$\rho_{mn}(z_0 + dz) = \sum_{s}^{N} \frac{1}{N} \langle m|dU_s|p\rangle \; \rho_{pq}(z_0) \; \langle q|dU_s^{\dagger}|n\rangle$$

$$|m\rangle = \int G_m(\mathbf{K}, z)|\mathbf{K}\rangle \; \frac{\mathrm{d}^2 K}{4\pi^2} \qquad\qquad \langle \mathbf{K}|m\rangle = G_m(\mathbf{K}, z)$$

Equation of motion in turbulence:

$$\nabla_T^2 g(\mathbf{x}) - i2k\partial_z g(\mathbf{x}) + 2k^2 \tilde{n}(\mathbf{x}) g(\mathbf{x}) = 0$$

In (transverse) Fourier domain:

$$-|\mathbf{K}|^2 G(\mathbf{K}, z) - i2k\partial_z G(\mathbf{K}, z) + 2k^2 N(\mathbf{K}) \star G(\mathbf{K}, z) = 0$$

$$G(\mathbf{K}, z_0 + dz) = G(\mathbf{K}, z_0) + \frac{idz}{2k} \left[|\mathbf{K}|^2 G(\mathbf{K}, z_0) - 2k^2 N(\mathbf{K}) \star G(\mathbf{K}, z_0) \right]$$

First order toward IPE

If $G(\mathbf{K}, z_0) = G_m(\mathbf{K}, z_0)$ then $G(\mathbf{K}, z_0 + dz) \neq G_m(\mathbf{K}, z_0 + dz)$ due to noise term

$$\begin{aligned} \langle m|dU_s|p\rangle &= \delta_{mp} + \frac{idz}{2k} \int G_m^*(\mathbf{K}, z_0) \left[|\mathbf{K}|^2 G_p(\mathbf{K}, z_0) \right. \\ & \left. -2k^2 N_s(\mathbf{K}, z_0) \star G_p(\mathbf{K}, z_0) \right] \frac{\mathrm{d}^2 K}{4\pi^2} \\ &= \delta_{mp} + idz\, \mathcal{P}_{mp} + dz\, \mathcal{L}_{s,mp} \end{aligned}$$

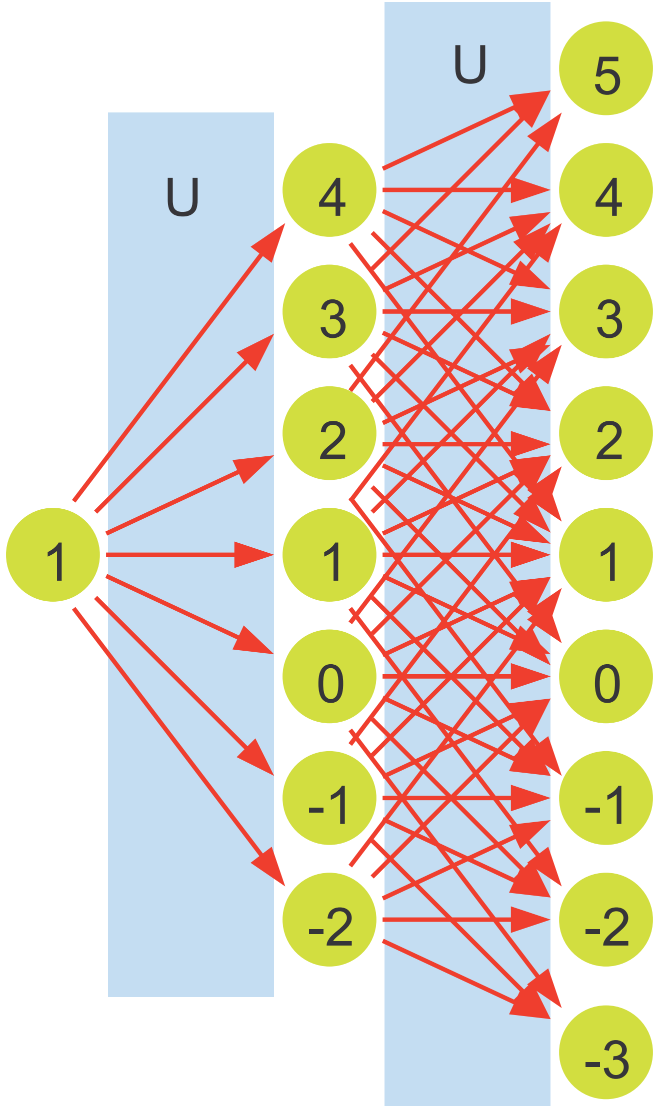
Density matrix elements:

$$\begin{aligned} \rho_{mn}(z_0 + dz) &= \rho_{mn}(z_0) + idz\, [\mathcal{P}, \rho(z_0)]_{mn} \\ & +dz \sum_s^N \frac{1}{N} \left[\mathcal{L}_{s,mp} \rho_{pn}(z_0) + \rho_{mq}(z_0) \mathcal{L}^\dagger_{s,qn} \right] \end{aligned}$$

Need to go to 2nd order for ensemble averages $\rightarrow$ (?) IPE in Lindblad form

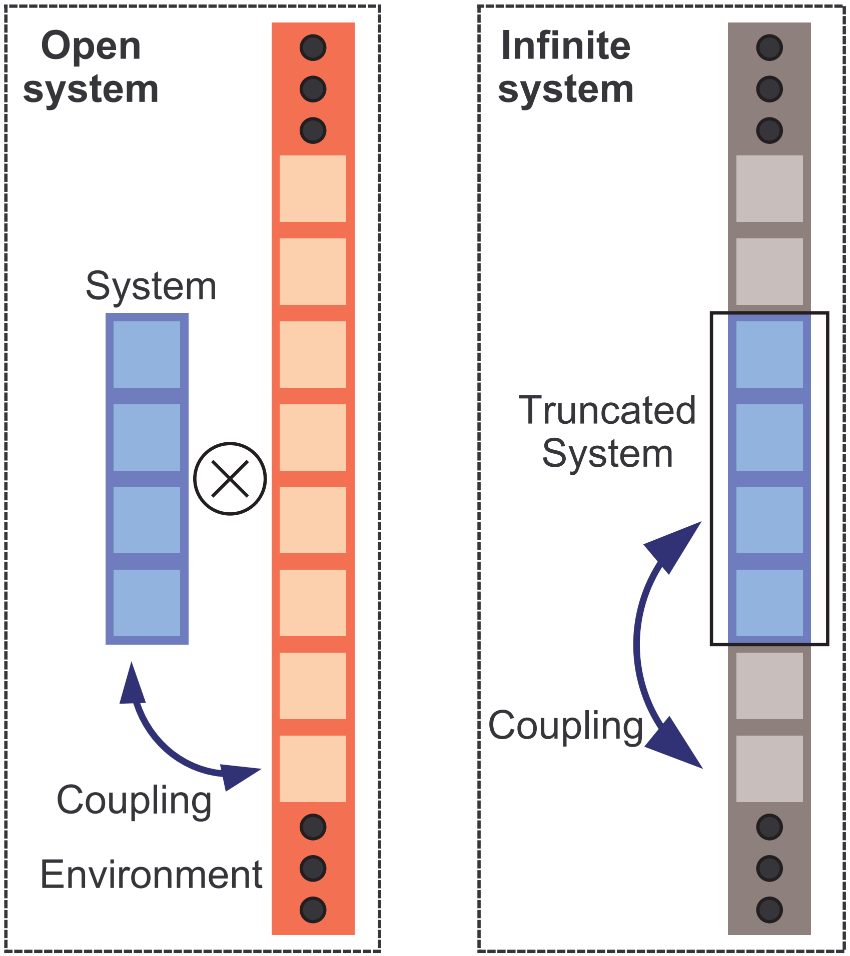
Truncating the IPE

- ▷ Modes couple to other modes due to scintillation
- ▷ Unitary process: Coupling coefficients are elements of unitary matrix
- ▷ Neigbouring modes have stronger couple than modes further apart
- ▷ Repeated process $\rightarrow$ backward coupling (perhaps neglectable?)



Truncation vs open system

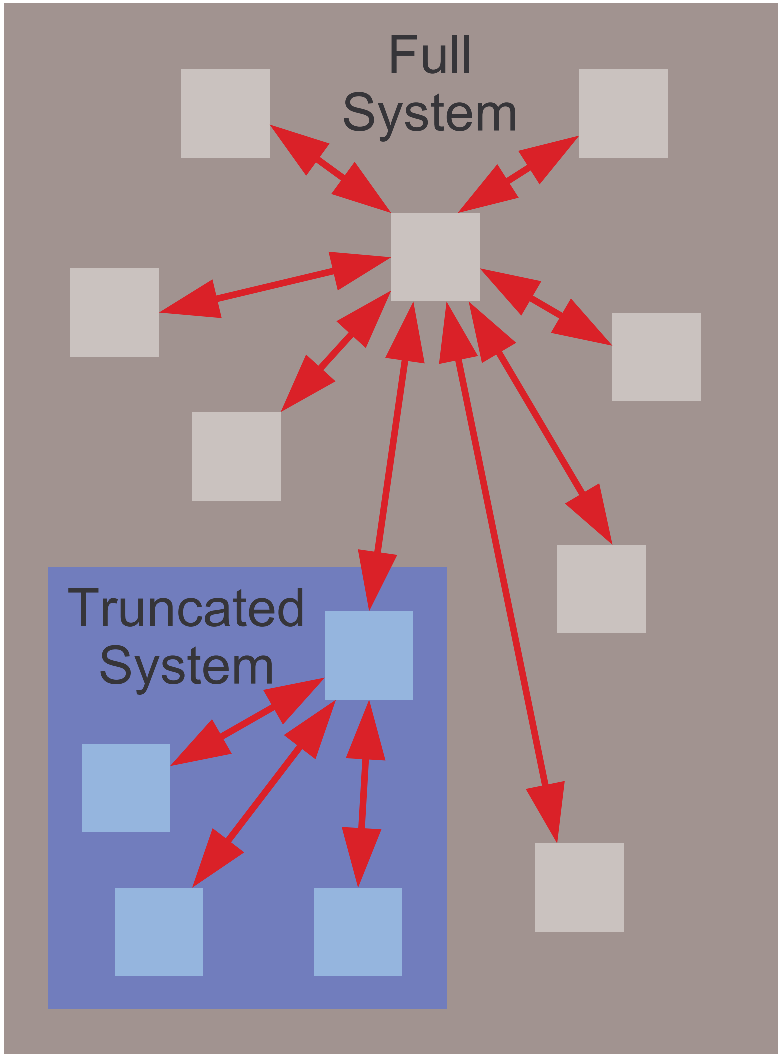
- Open system case:
  Information flow
  from system to
  environment
- Truncated case:
  Information flow
  from lower order modes
  to higher order modes
- Full system:
  Unitary process
- Truncated system:
  NOT unitary
  (‘sub’-unitary)

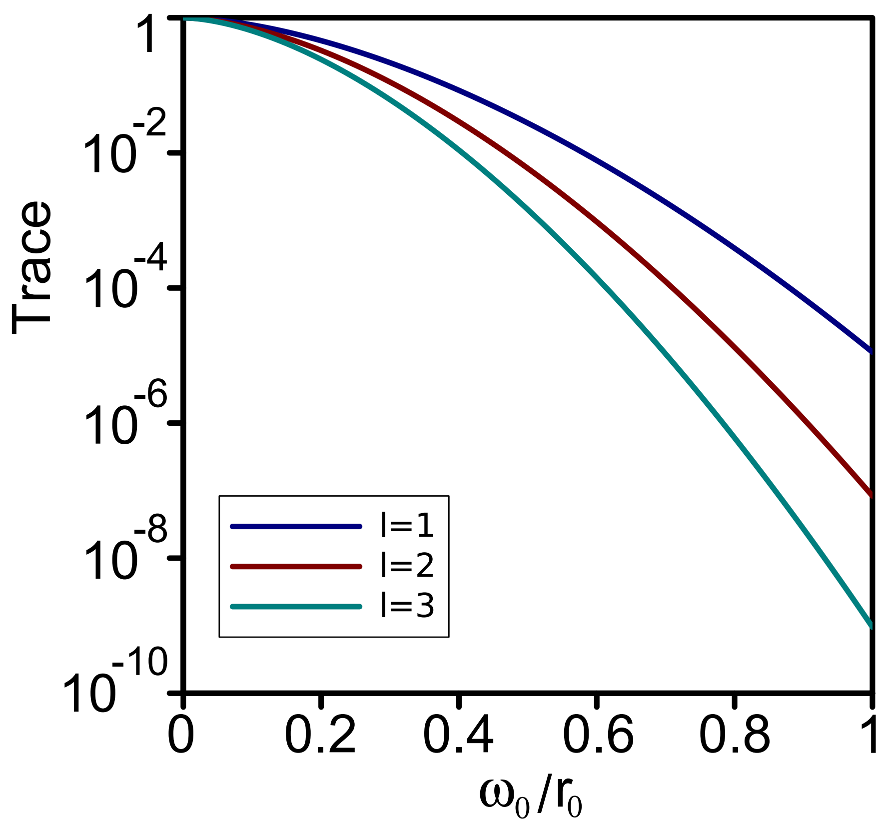


IPE is a combination of both

Inter- and intra-modal coupling

Do we need to be concerned about backward coupling?





$\Rightarrow$ backward coupling is minimal

State of the union for the IPE

Can we trust the IPE?

- ▷ Not trace preserving $\rightarrow$ can renormalize
- ▷ Hermitian
- ▷ Positivity $\rightarrow$ fighting chance
  Needs more work
- ▷ Truncation $\rightarrow$ backward coupling is small
- ▷ Predictions do not agree with Paterson model
  Which one is correct?
  $\rightarrow$ experimental measurements and numerical simulations

Robust states

Are the maximally entangled states also the ones that will retain most entanglement after propagation through turbulence? If not, what are the most robust states?

- ▷ Solve IPE for:
  - — symmetric and asymmetric qubits (2D)
  - — symmetric qutrits (3D)
- ▷ Arbitrary initial pure state
- ▷ Consider decoherence of:
  - — Bell states (2D subspace in qutrit states)
  - — Maximally entangled qutrit states
  - — Optimized qutrit states
- ▷ Within Rytov limit (small distances)

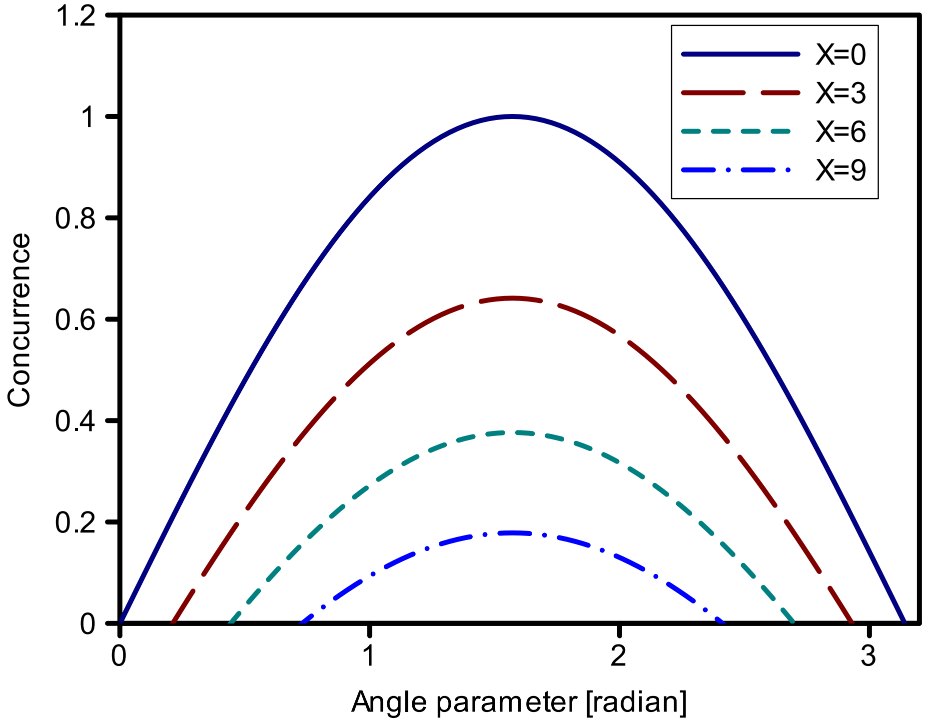
Symmetric qubit

For symmetric qubit:

$$|\psi\rangle = \cos(\phi/2)\exp(i\alpha)|1,1\rangle + \sin(\phi/2)\exp(-i\alpha)|-1,-1\rangle$$

Normalized propagation distance:

$$X = \frac{54.1 z w_0^{5/3} C_n^2}{\lambda^2}$$



Independent of α

Violates factorization law[a]

[a] T Konrad, et al., Nature Physics, **4**, 99 (2008)

Asymmetric qubit

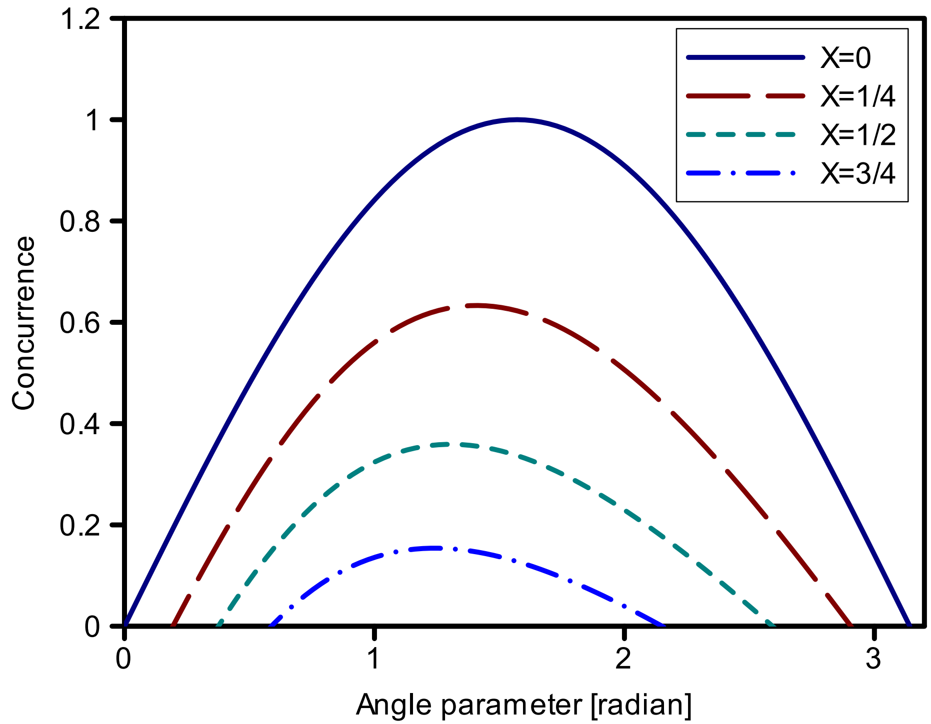
For asymmetric qubit:

$$|\psi\rangle = \cos(\phi/2)\exp(i\alpha)|1,1\rangle + \sin(\phi/2)\exp(-i\alpha)|0,0\rangle$$

Normalized propagation distance: X

Asymmetric in inter-modal coupling

Decays quicker due to stronger intra-modal coupling



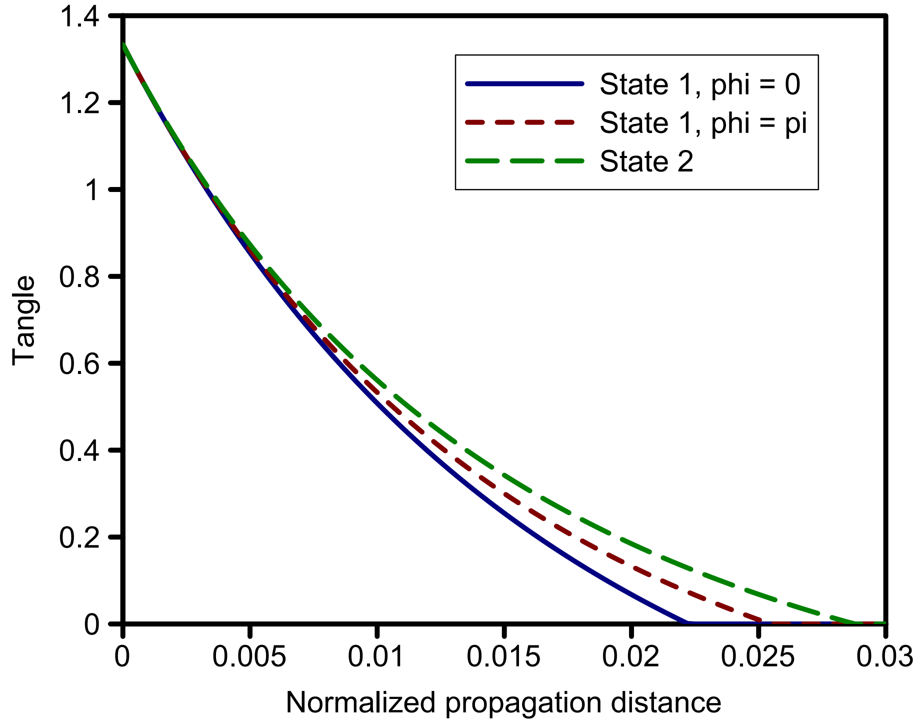
Maximally entangled qutrits

Tangle: $\tau = 2\,\mathrm{tr}\{\rho^2\} - \mathrm{tr}\{\rho_1^2\} - \mathrm{tr}\{\rho_2^2\}$
Theoretical maximum for qutrits: $\tau = 4/3$

State 1 ($\phi = 0$):
$|1,1\rangle + |0,0\rangle + |-1,-1\rangle$

State 1 ($\phi = \pi$):
$|1,1\rangle + i|0,0\rangle + |-1,-1\rangle$

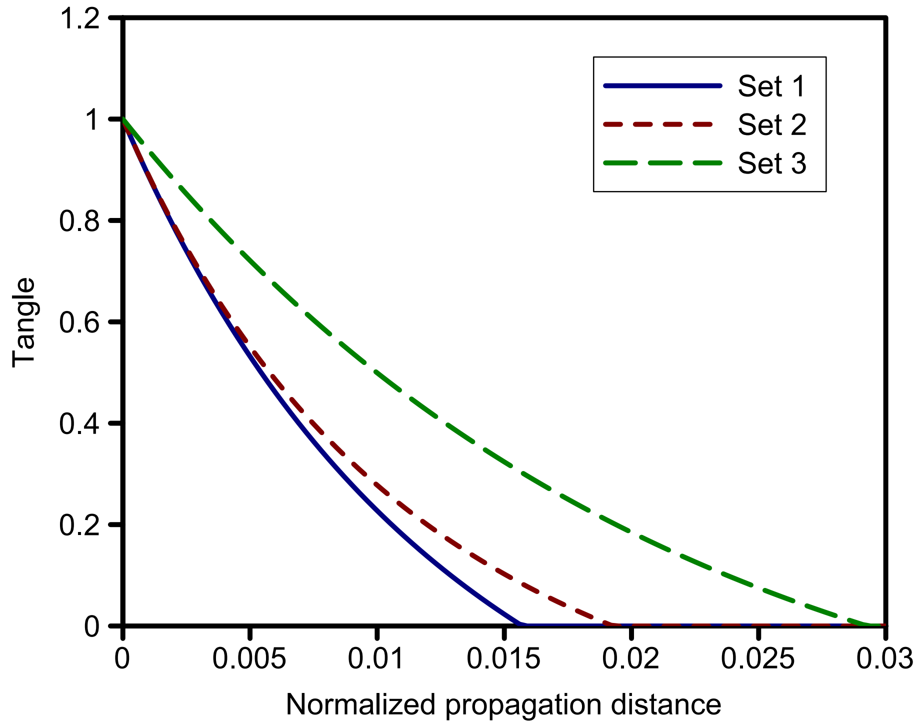
State 2:
$|1,1\rangle + |0,-1\rangle + |-1,0\rangle$



Bell states

Three sets:
Set 1: $\{|\pm1,\pm1\rangle \pm |0,0\rangle\}$
Set 2: $\{|\pm1,0\rangle \pm |0,\pm1\rangle\}$
Set 3: $\{|1,-1\rangle \pm |-1,1\rangle, |1,1\rangle \pm |-1,-1\rangle\}$

Decay at different rates due to difference in intra-modal coupling strengths

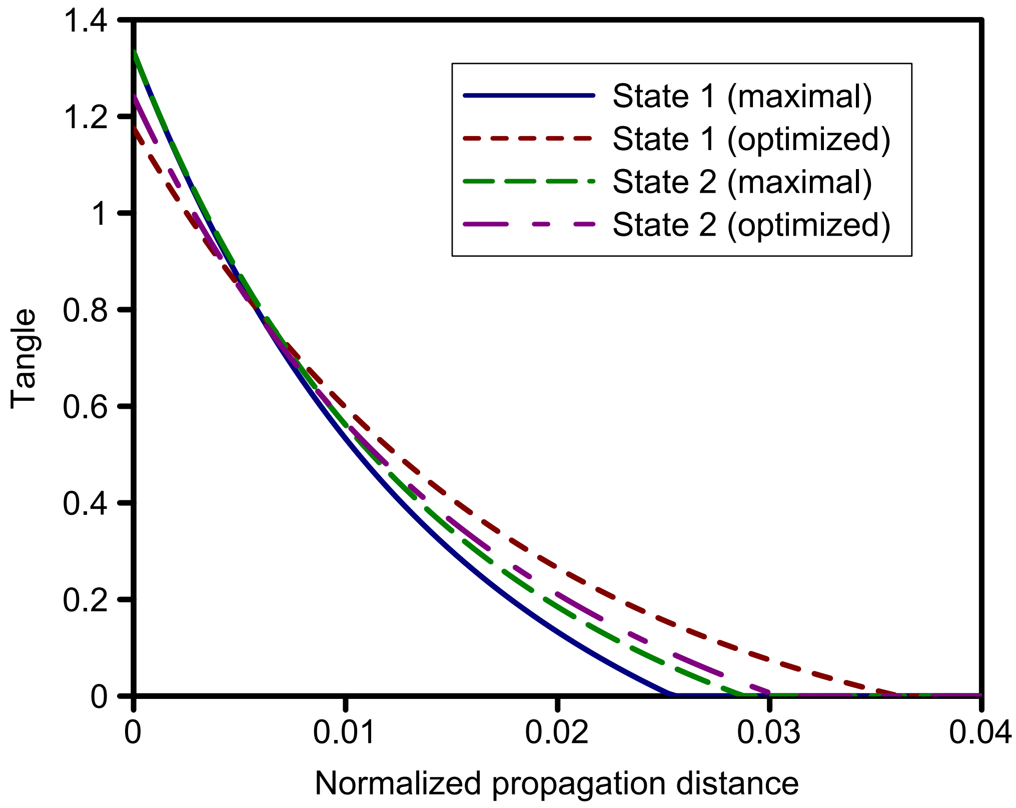


Optimized qutrits

Use additional parameters for relative weighting of terms

State 1: $\cos(\beta)|1,1\rangle + \sin(\beta)\exp(i\phi)|0,0\rangle + \cos(\beta)|-1,-1\rangle$

State 2: $\sin(\beta)|1,1\rangle + \cos(\beta)|0,-1\rangle + \cos(\beta)|-1,0\rangle$

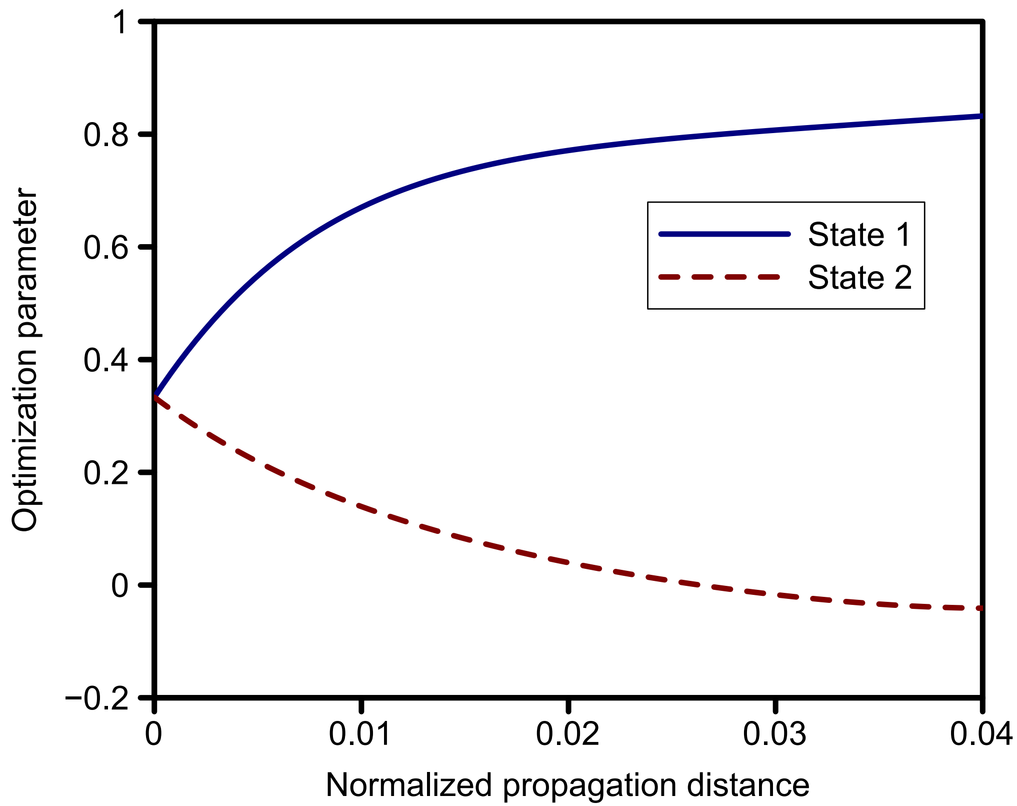


Optimized parameter

The optimization parameters
depend on the propagation distance

State 1: $\cos(\beta)|1,1\rangle + \sin(\beta)\exp(i\phi)|0,0\rangle + \cos(\beta)|-1,-1\rangle$
State 2: $\sin(\beta)|1,1\rangle + \cos(\beta)|0,-1\rangle + \cos(\beta)|-1,0\rangle$

$|\cos(2\beta)|$ as function of
propagation distance

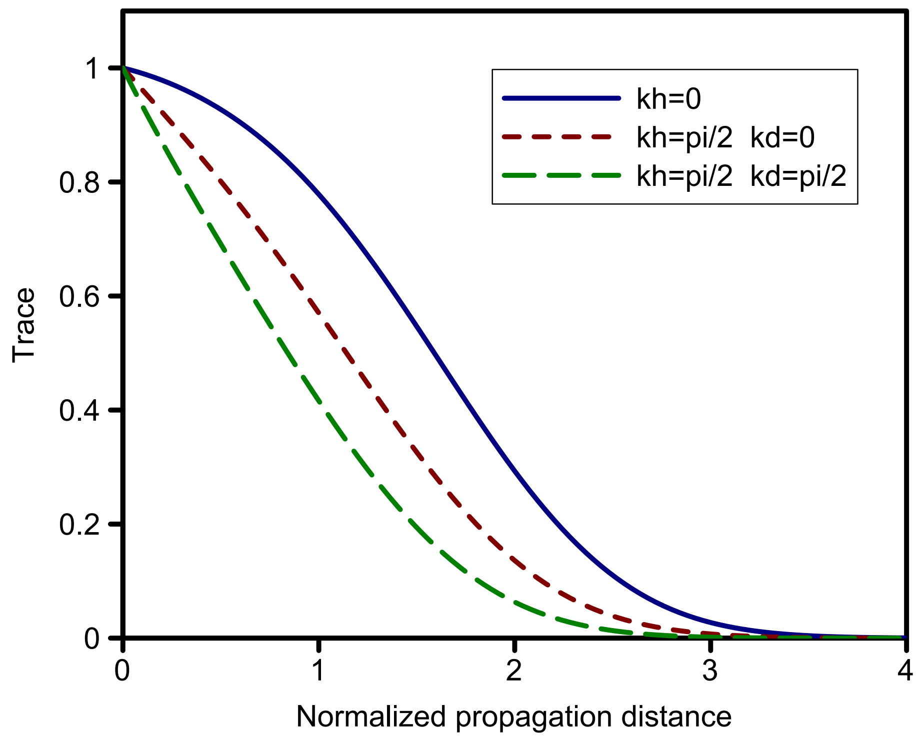


Optimized trace

Consider

$$|\psi\rangle = \cos(kh)|0,0\rangle + \sin(kh)\left[\cos(kd)|\pm 1, \pm 1\rangle + \sin(kd)|0, \pm 1\rangle\right]$$

One can optimize the trace but ....



Optimized trace implies NO entanglement

Trace and entanglement work against each other

Conclusions

- ▷ Decoherence in turbulence needs equation in z
- ▷ Paterson model — single phase screen
  - — Concurrence decays to zero in weak to moderate turbulence
  - — Quadratic structure function approximation
  - — Higher order OAM beyond weak limit
- ▷ IPE — could be OK but
  - — Positivity still needs confirmation
  - — Effect of truncation
- ▷ Possible to improve robustness
- ▷ Trace decays sooner than entanglement
  $\rightarrow$ Could be the most serious issue for quantum communication