Evolution equation for classical and quantum light in turbulence

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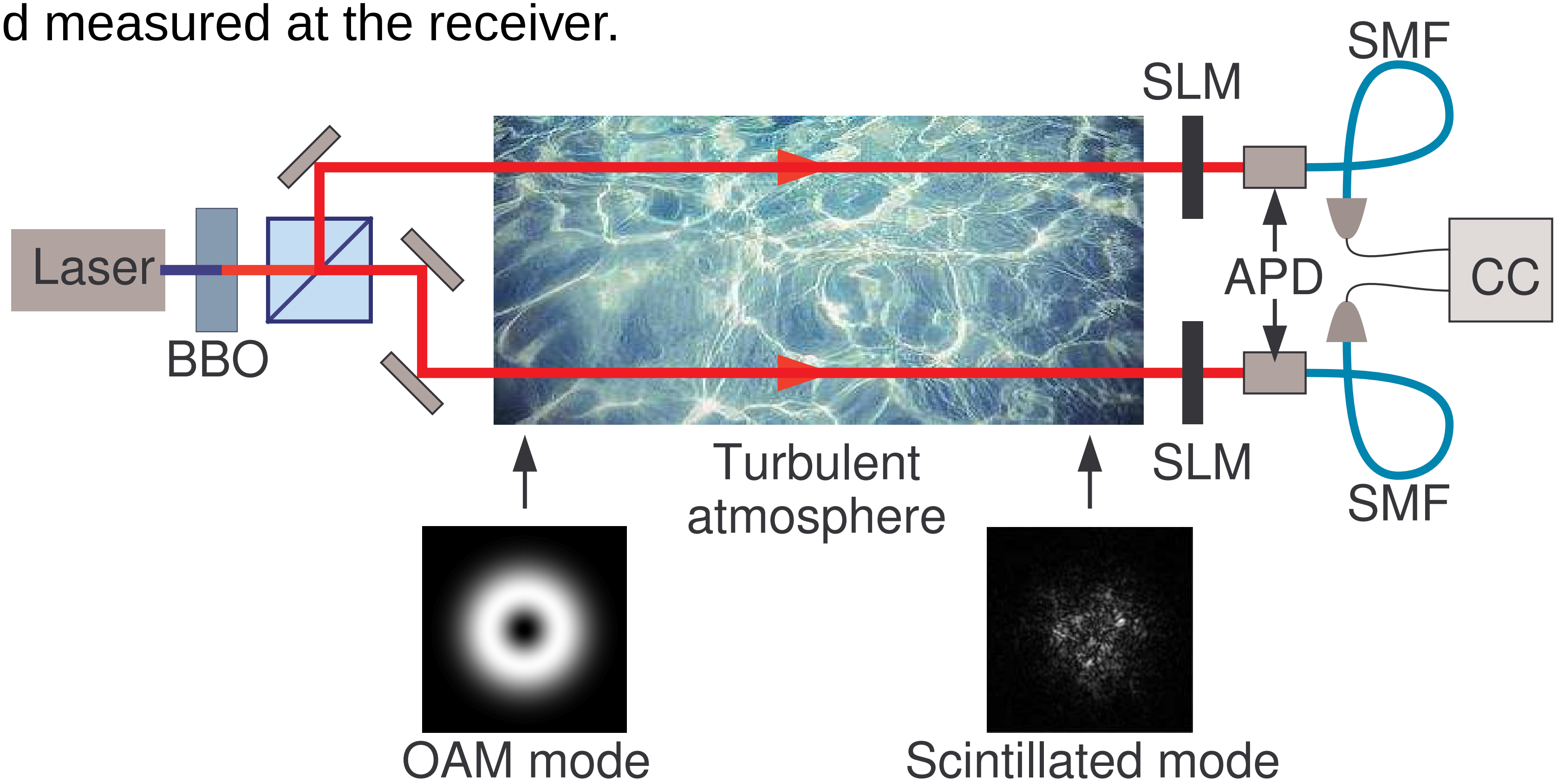
Presented at the
Propagation through and Characterization of Distributed Volume Turbulence and Atmospheric Phenomena Conference,
Renaissance Arlington Capital View Hotel, Arlington, Virginia, USA
8 June 2015

our future through science

Free-space quantum communication

The entangled photon pair is sent through the atmosphere and measured at the receiver.



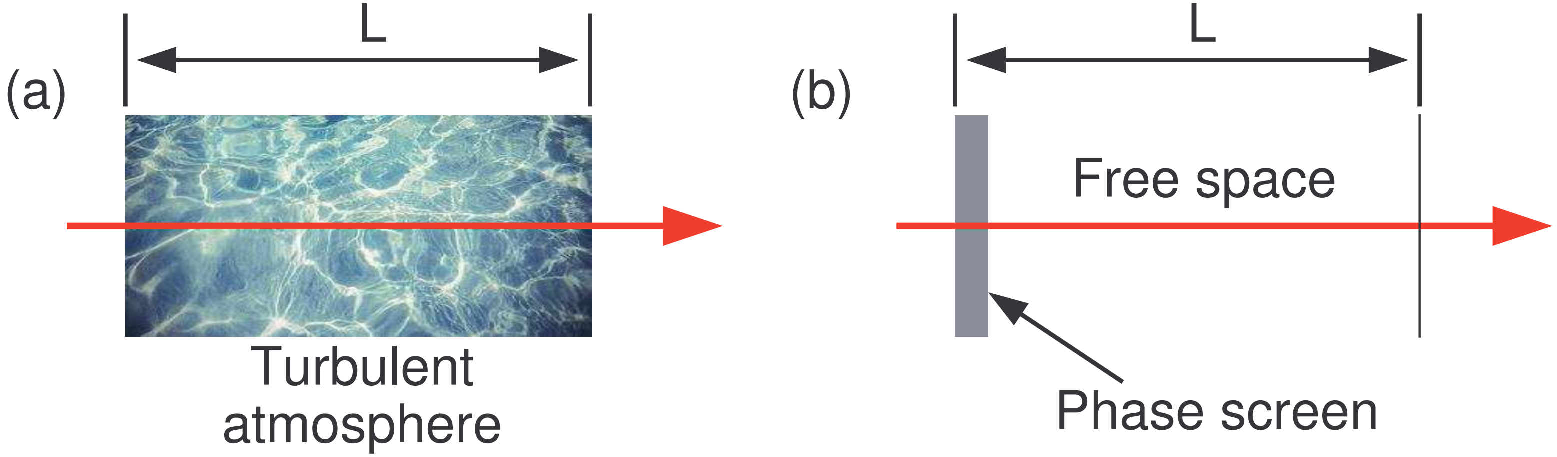
Turbulence distorts the OAM modes $\Rightarrow$ loss of entanglement

$\Rightarrow$ How to determine the evolution of a quantum state in turbulence?

Single phase screen

To investigate evolution of quantum states in turbulence

Single phase screen (SPS) approach (under weak scintillation): [a]



Ensemble averaged density matrix element:

$$\rho_{mn}(z) = \int E_m^*(\mathbf{r}_1) E_n(\mathbf{r}_2) \rho_0(\mathbf{r}_1, \mathbf{r}_2) \exp\left[-\frac{1}{2} D_\theta(\Delta r)\right] \mathrm{d}^2 r_1 \mathrm{d}^2 r_2$$

Modal basis functions: $E_n(\mathbf{r})$

Structure function: $D_\theta(\Delta r) = 6.88(|\mathbf{r}_1 - \mathbf{r}_2|/r_0)^{5/3}$

Fried parameter: $r_0 = 0.185(\lambda^2/C_n^2 z)^{3/5}$

C_n is the structure constant; λ is wavelength and z is propagation distance

[a] C. Paterson, Phys. Rev. Lett. 94, 153901 (2005).

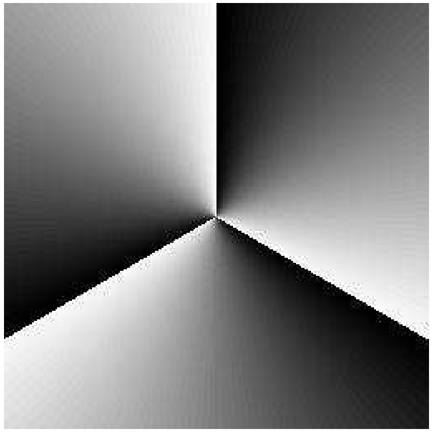
Laguerre-Gauss (LG) modes

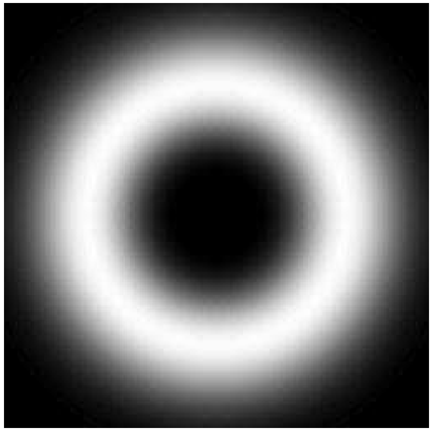
General solutions of the paraxial wave equation
in normalized polar coordinates:

$$M_{p,\ell}^{\mathrm{LG}}(r,\phi,t) = \mathcal{N}\frac{(1+it)^p}{(1-it)^{p+|\ell|+1}} r^{|\ell|} \exp(i\ell\phi) L_p^{|\ell|}\left(\frac{2r^2}{1+t^2}\right) \exp\left(\frac{-r^2}{1-it}\right)$$

$$r = \frac{\sqrt{x^2+y^2}}{w_0} \qquad t = \frac{z}{z_R} \qquad z_R = \frac{\pi w_0^2}{\lambda}$$

$$\mathcal{N} = \left[\frac{2^{1+|\ell|}p!}{\pi(p+|\ell|)!}\right]^{1/2}$$





$L_p^{|\ell|}$ — associate Laguerre polynomials
p — radial mode index (non-negative integer)
ℓ — azymuthal index (signed integer)

Orbital angular momentum (OAM) per photon $= \ell\hbar$.

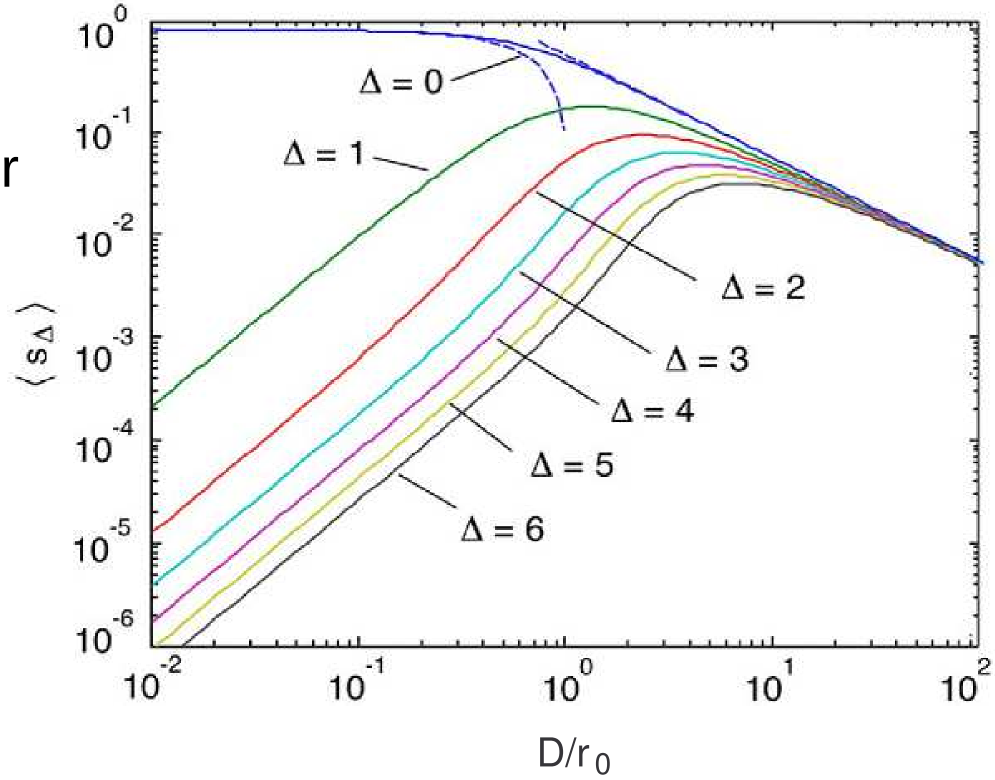
OAM in turbulence (SPS approach) I

Application of SPS approach:

Change in OAM after turbulence due to OAM scattering: [a]

$\langle s_\Delta \rangle$ — power fraction
D — receiving aperture diameter
r_0 — Fried parameter:

$$r_0 = 0.185 \left(\frac{\lambda^2}{C_n^2 z} \right)^{3/5}$$



[a] G. A. Tyler and R. W. Boyd, Opt. Lett., 34, 142 (2009)

OAM in turbulence (SPS approach) II

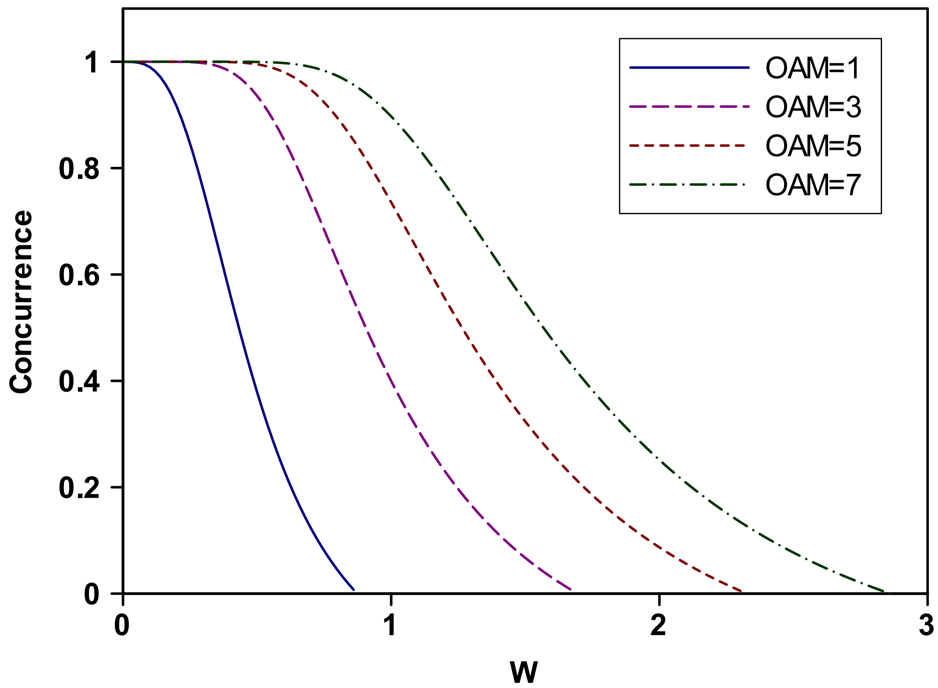
Application of SPS approach:

Decay of OAM entanglement after turbulence: [a]

Concurrence:
measure of qubit entanglement

$$\mathcal{W} = \frac{w_0}{r_0}$$

w_0 — beam radius
r_0 — Fried parameter



[a] B. J. Smith and M. G. Raymer, Phys. Rev. A, 74, 062104 (2006)

Weak scintillation limit

Consider scintillation strength vs normalized propagation distance

Rytov variance:

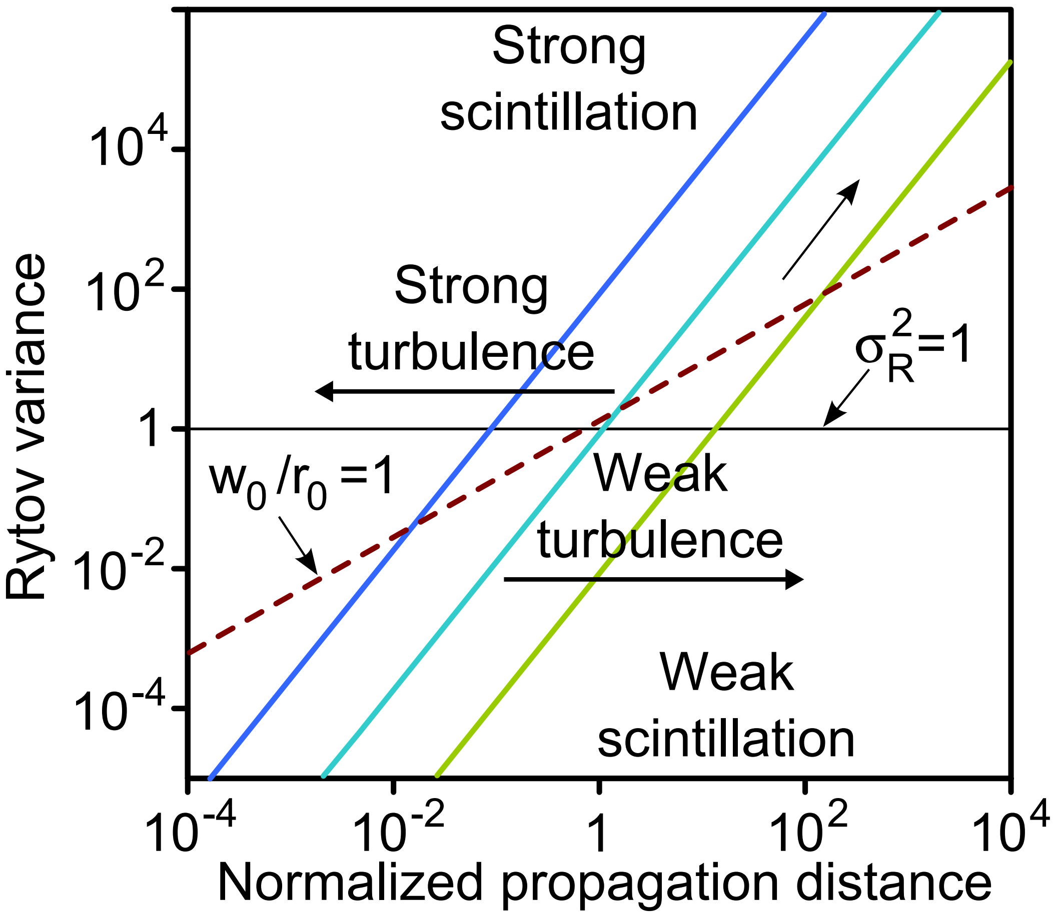
$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} z^{11/6}$$

i.t.o. $t = z/z_R = z\lambda/\pi w_0^2$:

$$\sigma_R^2 = \frac{85.6 C_n^2 w_0^{11/3} t^{11/6}}{\lambda^3}$$

and i.t.o. $\mathcal{W} = w_0/r_0$:

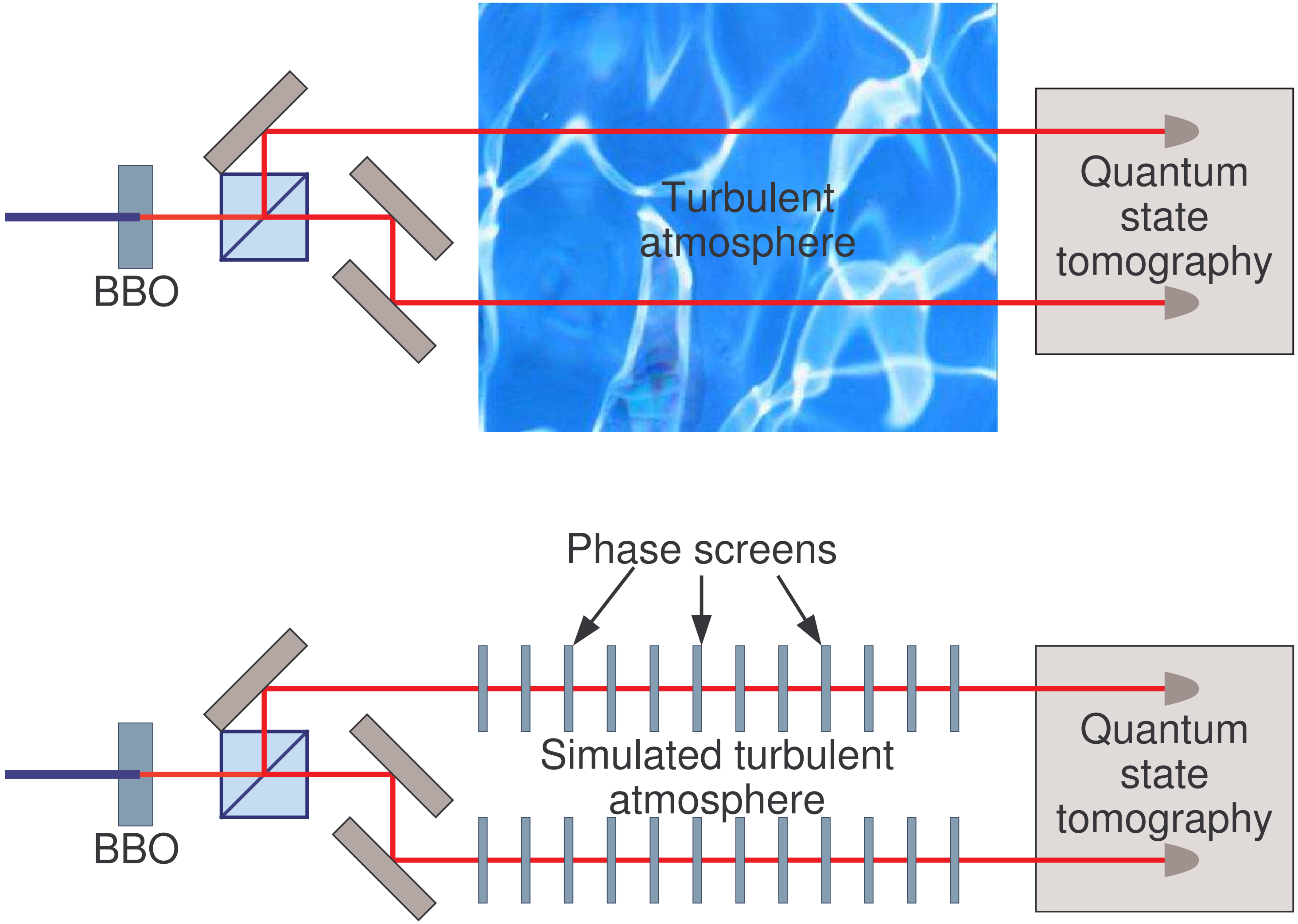
$$\sigma_R^2 = 1.64 \left(\frac{w_0}{r_0}\right)^{5/3} t^{5/6}$$



$\Rightarrow$ Weak scintillation for all $\mathcal{W}$ require strong turbulence $\mathcal{K}_0 \to \infty$.

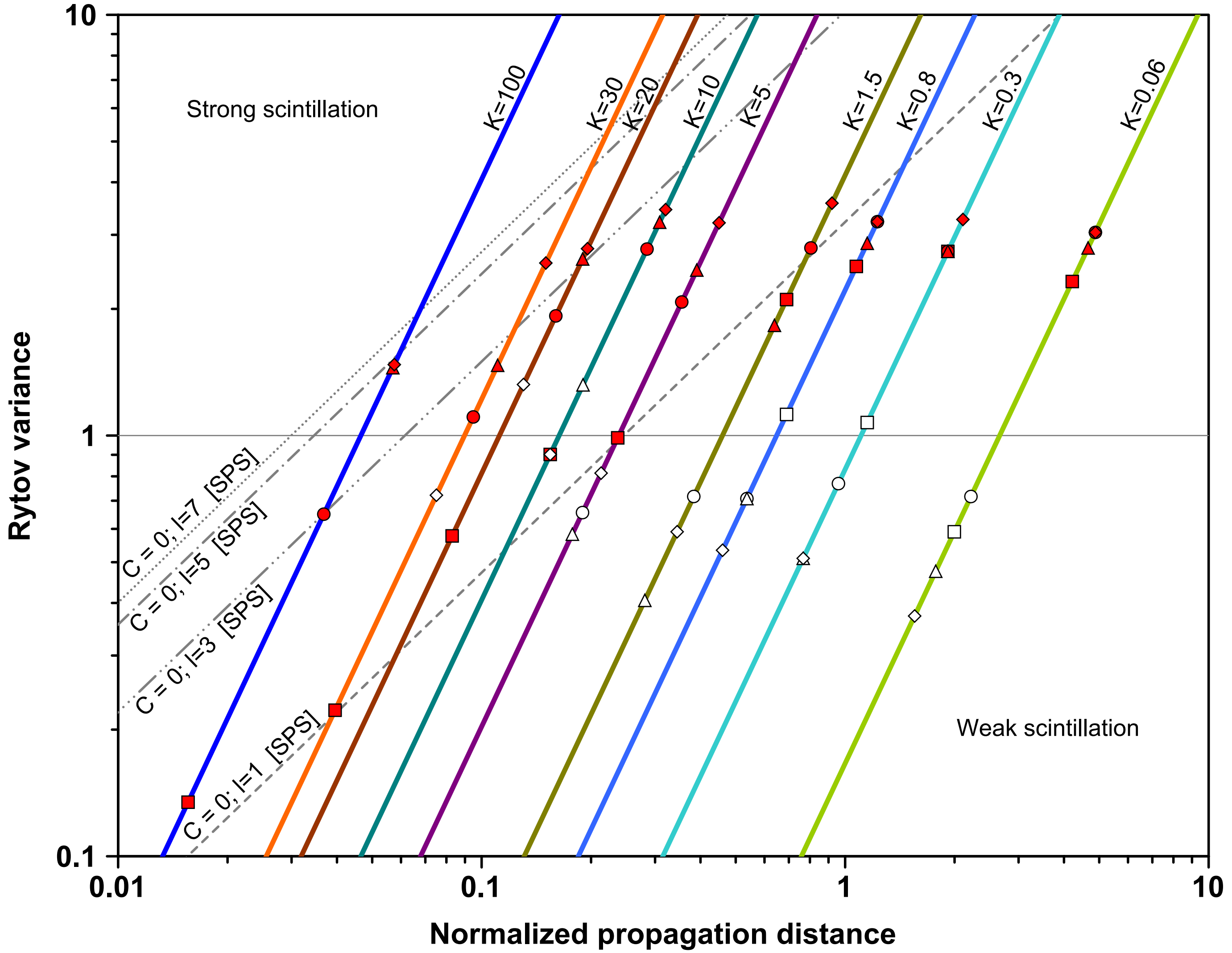
Multiple phase screen simulations

Simulate turbulent medium using multiple phase screens:



Numerical simulation results

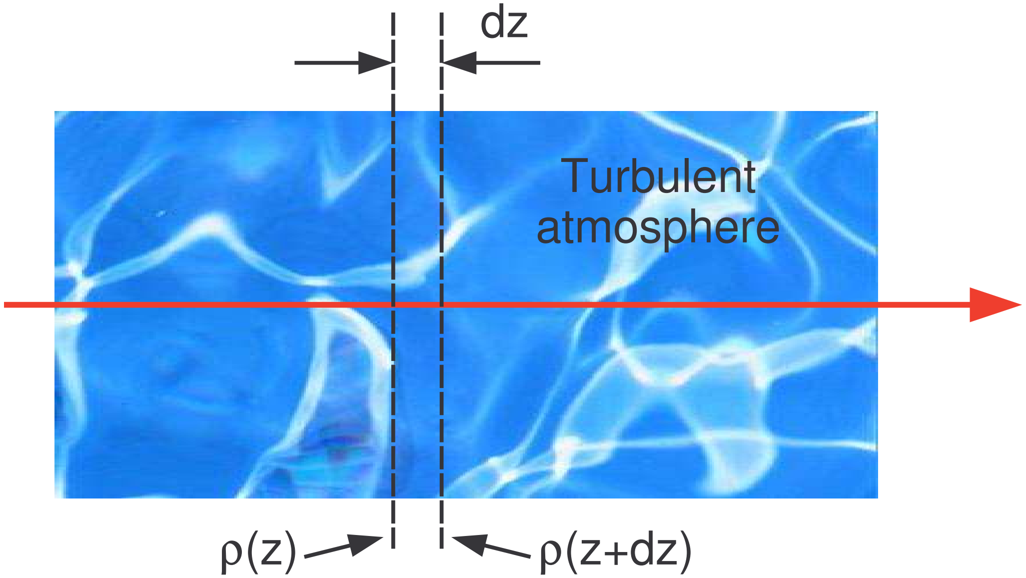
Entanglement 'sudden death' [a]



[a] A. Hamadou Ibrahim, *et al.*, Phys. Rev. A, 90, 052115 (2014)

Infinitesimal propagation (MPS approach)

Multiple phase screen (MPS) approach:



Infinitesimal propagation:
$\rho(z) \to \rho(z + \delta z)$

Using paraxial wave equation in an inhomogenous medium:

$$\nabla_T^2 g(\mathbf{x}) - i2k\partial_z g(\mathbf{x}) + 2k^2 \tilde{n}(\mathbf{x}) g(\mathbf{x}) = 0$$

In the (2D transverse spatial) Fourier domain:

$$G(\mathbf{a}, z + \delta z) = G(\mathbf{a}, z) + i\pi\lambda\delta z \left[|\mathbf{a}|^2 G(\mathbf{a}, z) - \frac{k^2}{2\pi^2} N(\mathbf{a}, z) \star G(\mathbf{a}, z) \right]$$

$G(\mathbf{a}, z)$ — transverse Fourier transformed field

$N(\mathbf{a}, z)$ — transverse Fourier transformed refractive index fluctuations

$\mathbf{a}$ — spatial frequency vector ($\mathbf{k} = 2\pi\mathbf{a}$)

Infinitesimal propagation equation

Expand to second order in fluctuations and evaluate ensemble average, using Markov approximation [$N(\mathbf{a}, z)$ is delta-correlated in z].

Infinitesimal propagation equation for a single photon: [a]

$$\begin{aligned} \partial_z \rho(\mathbf{a}_1, \mathbf{a}_2, z) &= i\pi\lambda\rho(\mathbf{a}_1, \mathbf{a}_2, z) \left(|\mathbf{a}_1|^2 - |\mathbf{a}_2|^2\right) \\ &\quad -k^2 \int \Phi_0(\mathbf{u}, 0) \left[\rho(\mathbf{a}_1, \mathbf{a}_2, z) - \rho(\mathbf{a}_1 - \mathbf{u}, \mathbf{a}_2 - \mathbf{u}, z)\right] \, \mathrm{d}^2 u \end{aligned}$$

$\rho(\mathbf{a}_1, \mathbf{a}_2, z)$ — single photon density function in plane wave basis
$\Phi_0(\mathbf{u}, 0)$ — power spectral density for the refractive index fluctuations

$\rightarrow$ Reminiscent of evolution of mutual coherence function:

Single photon density function $\leftrightarrow$ mutual coherence function

[a] F. S. Roux, Phys. Rev. A, 83, 053822 (2011);

—, J. Phys. A: Math. Theor., 47, 195302 (2014).

Photon pairs (entanglement)

Infinitesimal propagation equation for photon pairs:

$$\begin{aligned}
\partial_z \rho(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, z) \quad &= \quad i\pi\lambda\rho(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, z)(|\mathbf{a}_1|^2 - |\mathbf{a}_2|^2 + |\mathbf{a}_3|^2 - |\mathbf{a}_4|^2) \\
&\quad -k^2 \int \Phi_0(\mathbf{u}, 0)\,[2\rho(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, z) \\
&\quad -\rho(\mathbf{a}_1 - \mathbf{u}, \mathbf{a}_2 - \mathbf{u}, \mathbf{a}_3, \mathbf{a}_4, z) \\
&\quad -\rho(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 - \mathbf{u}, \mathbf{a}_4 - \mathbf{u}, z)]\ \mathrm{d}^2 u
\end{aligned}$$

$\rho(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, z)$ — photon pair density ‘matrix’ in plane wave basis

Density operator:

$$\hat{\rho} = \int |\mathbf{a}_1\rangle_A\, |\mathbf{a}_3\rangle_B\, \rho(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, z)\, \langle\mathbf{a}_2|_A\, \langle\mathbf{a}_4|_B\ \mathrm{d}^2 a_1\ \mathrm{d}^2 a_2\ \mathrm{d}^2 a_3\ \mathrm{d}^2 a_4$$

$\mathbf{a}_1$ and $\mathbf{a}_2$ — spatial frequencies for photon A

$\mathbf{a}_3$ and $\mathbf{a}_4$ — spatial frequencies for photon B

Solving the IPE — without truncation

1. Remove free-space term (quadratic phase factor):

$$\rho(\mathbf{a}_1, \mathbf{a}_2, z) = F(\mathbf{a}_1, \mathbf{a}_2, z) \exp[i\pi\lambda z(|\mathbf{a}_1|^2 - |\mathbf{a}_2|^2)]$$

2. Redefine coordinates i.t.o. sums and differences: $\mathbf{a}_{1,2} = \mathbf{a}_s \pm \mathbf{a}_d/2$

   so that: $F(\mathbf{a}_1, \mathbf{a}_2, z) = F(\mathbf{a}_s + \mathbf{a}_d/2, \mathbf{a}_s - \mathbf{a}_d/2, z) \equiv G(\mathbf{a}_s, \mathbf{a}_d, z)$

3. Inverse Fourier transform w.r.t. sum coordinates:

$$H(\mathbf{x}, \mathbf{a}_d, z) = \int G(\mathbf{a}_s, \mathbf{a}_d, z) \exp(-i2\pi\mathbf{a}_s \cdot \mathbf{x})\ \mathrm{d}^2 a_s$$

Resulting (single photon) equation:

$$\partial_z H(\mathbf{x}, \mathbf{a}_d, z) = -k^2 H(\mathbf{x}, \mathbf{a}_d, z) Q(\lambda z \mathbf{a}_d + \mathbf{x})$$

where: $$Q(\mathbf{x}) = \int \Phi_0(\mathbf{u}, 0)\,[1 - \exp(-i2\pi\mathbf{x} \cdot \mathbf{u})]\ \mathrm{d}^2 u$$

Solution: $$H(\mathbf{x}, \mathbf{a}_d, z) = H_0(\mathbf{x}, \mathbf{a}_d) \exp\left[-k^2 \int_0^z Q(\lambda z' \mathbf{a}_d + \mathbf{x})\ \mathrm{d}z'\right]$$

Kolmogorov

Consider Q-integral:

$$Q(\mathbf{x}) = \int \Phi_0(\mathbf{u}, 0) \left[1 - \exp(-i2\pi \mathbf{x} \cdot \mathbf{u})\right] \, \mathrm{d}^2 u$$

For Kolmogorov:
$$\Phi_0(\mathbf{u}, 0) = \frac{0.033 C_n^2}{(2\pi)^{2/3} |\mathbf{u}|^{11/3}}$$

(2π-factor due to use of spatial frequency)

Result: $Q(\mathbf{x}) = 1.457 C_n^2 |\mathbf{x}|^{5/3}$

Hence,

$$H(\mathbf{x}, \mathbf{a}_d, z) = H_0(\mathbf{x}, \mathbf{a}_d) \exp\left[-1.457 C_n^2 k^2 \int_0^z |\lambda z' \mathbf{a}_d + \mathbf{x}|^{5/3} \, \mathrm{d}z'\right]$$

Quadratic structure function

Hard to evaluate integrals with power of $5/3$.

Quadratic structure function approximation: $5/3 \to 2$
(but beware of anomalous dimension)

The result after z-integration:

$$H(\mathbf{x}, \mathbf{a}_d, z) = H_0(\mathbf{x}, \mathbf{a}_d) \exp\left[\frac{-2\mathcal{K}_0\Theta}{w_0^3} p(\mathbf{x}, \mathbf{a}_d, z)\right]$$

where
$\Theta = \lambda/\pi w_0$ — beam divergence angle
w_0 — beam radius

$$\mathcal{K}_0 = 2.9\frac{\pi^3 C_n^2 w_0^{11/3}}{\lambda^3} = 2.9\kappa$$

$$p(\mathbf{x}, \mathbf{a}_d, z) = |\mathbf{x}|^2 z + \mathbf{a}_d \cdot \mathbf{x}\, \lambda z^2 + \frac{1}{3}|\mathbf{a}_d|^2 \lambda^2 z^3$$

Single photon state

Convert back to original variables

Evolving (single photon) state:

$$\begin{aligned}\rho(\mathbf{a}_1, \mathbf{a}_2, t) &= \frac{\pi w_0^2}{2\mathcal{K}_0 t} \int \rho_0(\mathbf{u}+\mathbf{a}_1, \mathbf{u}+\mathbf{a}_2) \exp\left\{-\pi^2 w_0^2 \left[\frac{\mathcal{K}_0 t^3}{6} |\mathbf{a}_1 - \mathbf{a}_2|^2\right.\right. \\ &\left.\left. -\mathrm{i}t\ \mathbf{u}\cdot(\mathbf{a}_1 - \mathbf{a}_2) - \mathrm{i}t\ (|\mathbf{a}_1|^2 - |\mathbf{a}_2|^2) + \frac{|\mathbf{u}|^2}{2\mathcal{K}_0 t}\right]\right\}\ \mathrm{d}^2u\end{aligned}$$

where

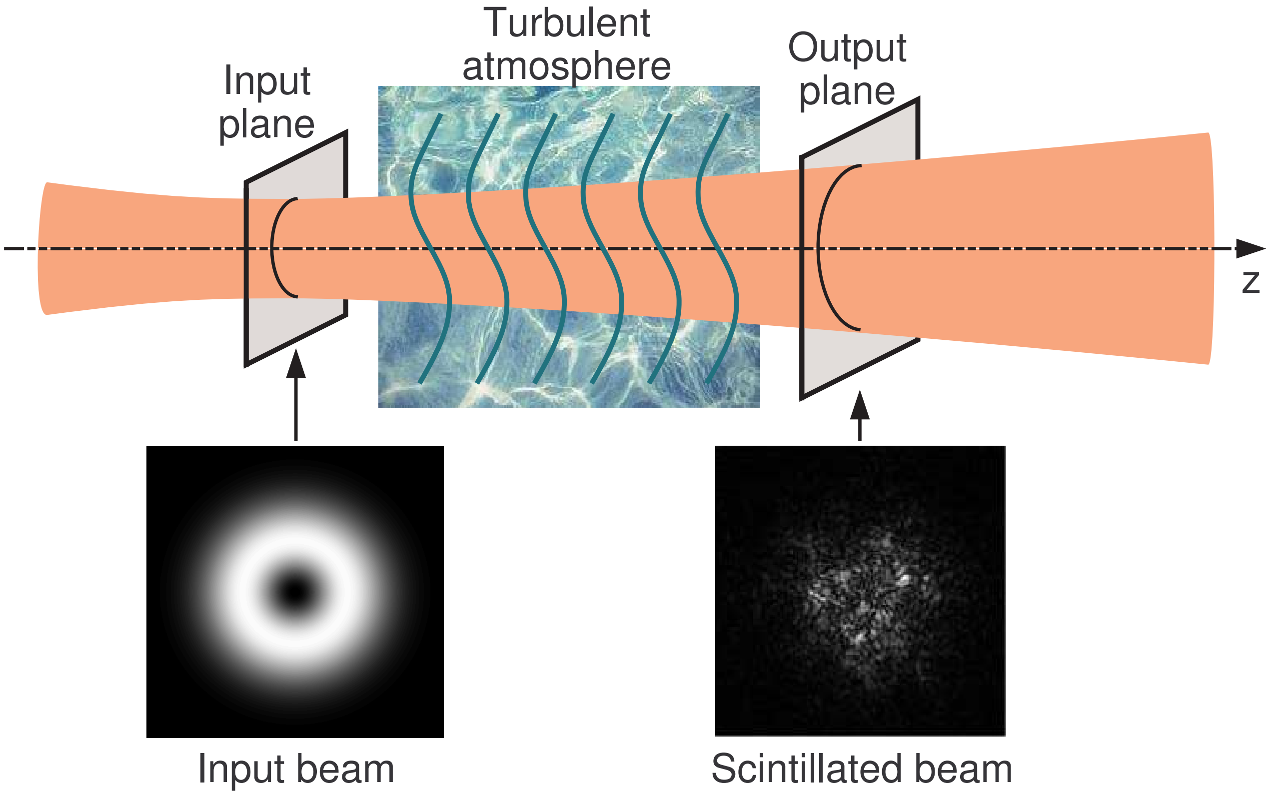
$t = z/z_R = z\lambda/\pi w_0^2$ — normalized propagation distance

w_0 — beam radius

$\rightarrow$ generalizable for two-photon states

Example: OAM scattering

As example, consider OAM scattering due to turbulence:



Input: $\rho_0(\mathbf{a}_1, \mathbf{a}_2) = M_{0,\ell}^{\rm LG}(\mathbf{a}_1) M_{0,\ell}^{\rm LG*}(\mathbf{a}_2)$ — (mutual coherence function)

$M_{p,\ell}^{\rm LG}(\mathbf{a})$ — angular spectrum of Laguerre-Gauss mode

p, ℓ — radial and azimuthal mode index

LG mode generating function

Represent angular spectra of LG modes i.t.o. generating function:

$$\mathcal{G} = \frac{\pi w_0}{1+\eta} \exp\left[\frac{i\pi(a \pm ib)w_0\mu - \pi^2(a^2+b^2)w_0^2\Omega(t,\eta)}{1+\eta}\right]$$

where (η, μ) are generating parameters for (p, ℓ),
and $\Omega(t,\eta) = 1 - it - (1+it)\eta$

To generate a particular LG mode's angular spectrum:

$$M_{p,\ell}^{\mathrm{LG}}(\mathbf{a}) = \mathcal{N}\left[\frac{1}{p!}\partial_\eta^p \partial_\mu^{|\ell|}\mathcal{G}\right]_{\eta,\mu=0}$$

where

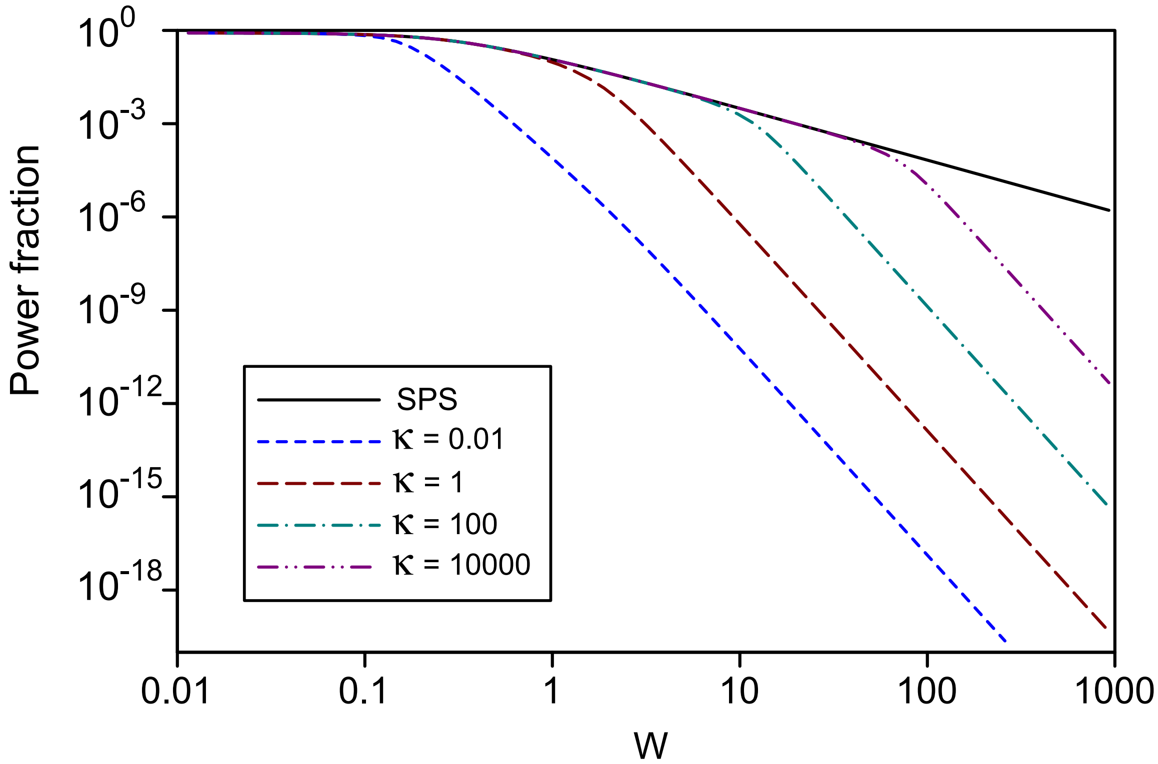
$$\mathcal{N} = \left[\frac{2^{1+|\ell|}p!}{\pi(p+|\ell|)!}\right]^{1/2}$$

Power fraction in LG basis: SPS vs MPS

Power fraction $s_\Delta = |\eta(t)|^2$ is square of coupling coefficient:

$$|\eta(t)|^2 = \int \rho(\mathbf{a}_1, \mathbf{a}_2, t) M_{0,\ell}^{\mathrm{LG}*}(\mathbf{a}_1, t) M_{0,\ell}^{\mathrm{LG}}(\mathbf{a}_2, t)\ d^2a_1\ d^2a_2.$$

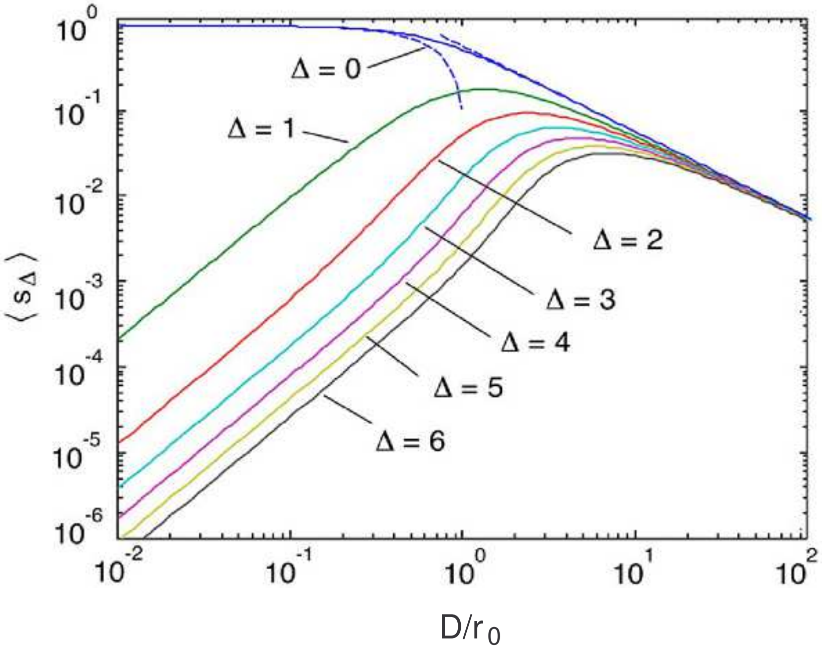
Single phase screen approach compared with multiple phase screen approach ($\kappa = 0.344\mathcal{K}_0$; input $\ell = 1$):



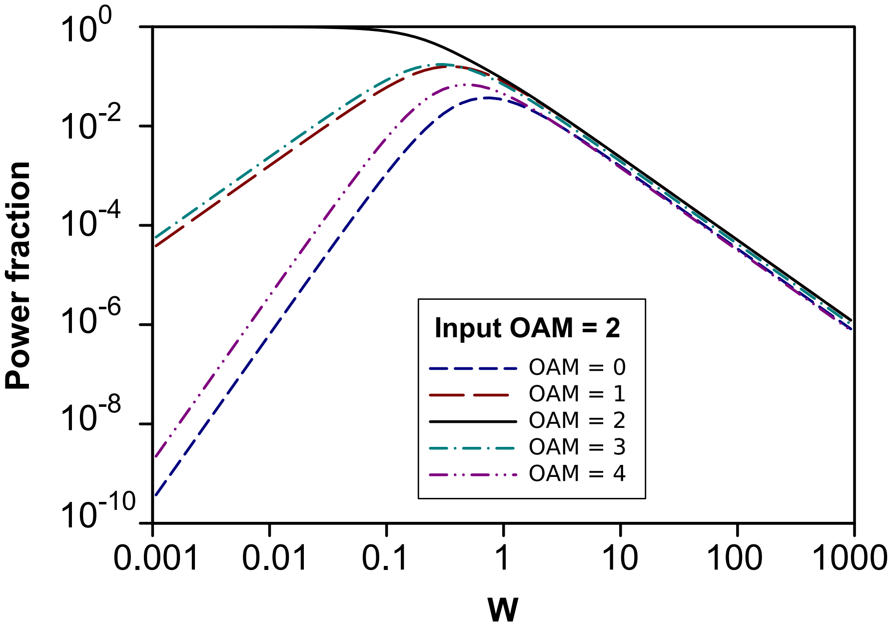
OAM scattering: weak scintillation limit

In weak scintillation limit (assume $p = 0$):

Previous analysis [a]



IPE with $\mathcal{K}_0 \to \infty$

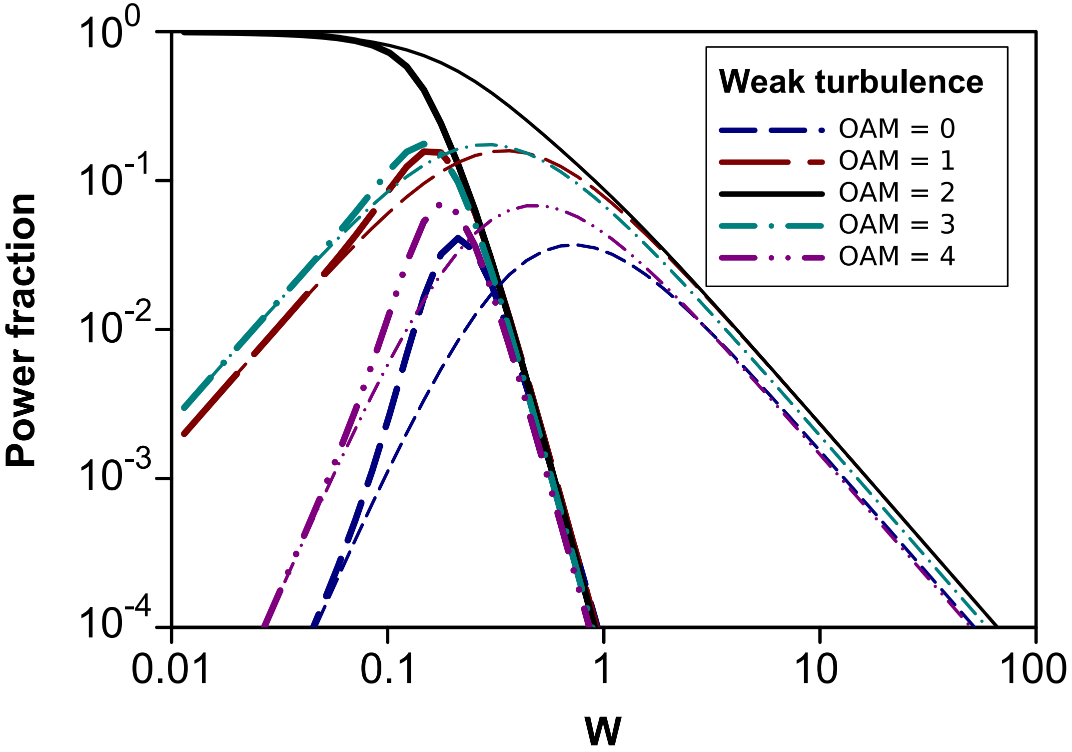


[a] G. A. Tyler and R. W. Boyd, Opt. Lett., 34, 142 (2009)

OAM scattering: weak turbulence I

Weak turbulence implies further propagation, entering the region of strong scintillation

Comparing IPE and SPS for $\kappa = 0.01$:

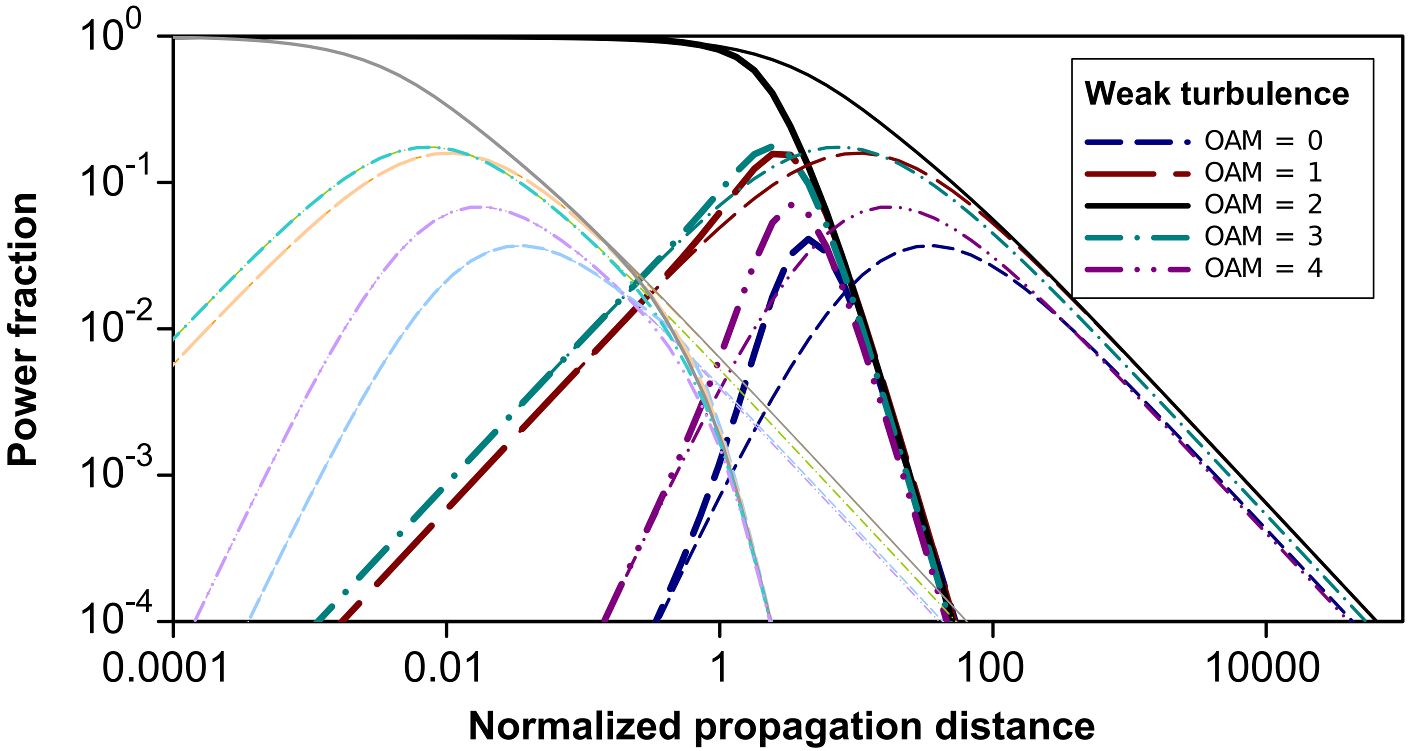


Scale in $\mathcal{W} = w_0/r_0$ sets in at the onset of strong scintillation

OAM scattering: weak turbulence II

The effect of the onset of strong scintillation
is seen at larger propagation distances.

Compare $\kappa = 0.01$ and $\kappa = 10$:



Strength of turbulence produce scaling in t for SPS
but with IPE there are other deviations

OAM scattering: partial coherence

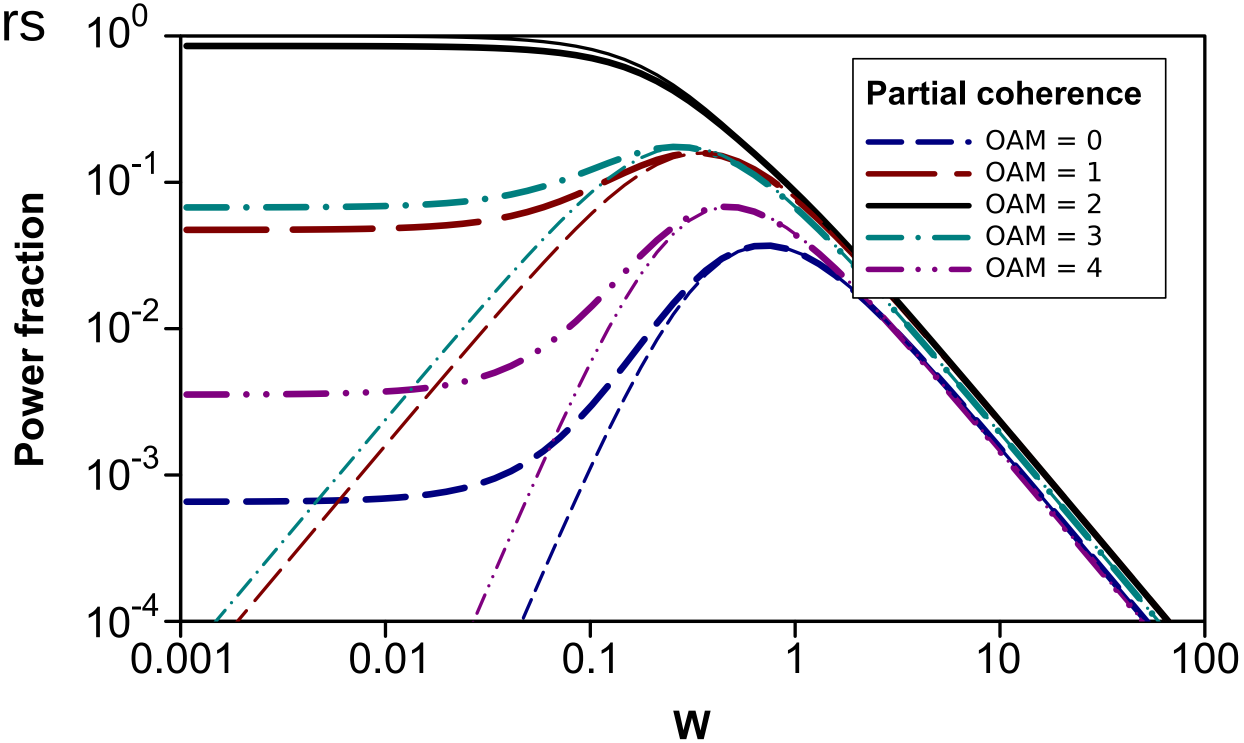
To consider an input state that is only partially coherent,
we modify the input state as follows:

$$\rho_0(\mathbf{a}_1, \mathbf{a}_2) = 2\pi\delta^2 \int \exp(-2\pi^2|\mathbf{u}|^2\delta^2) M_{0,\ell}^{\mathrm{LG}}(\mathbf{a}_1 - \mathbf{u}) M_{0,\ell}^{\mathrm{LG}*}(\mathbf{a}_2 - \mathbf{u})\ \mathrm{d}^2 u$$

where δ represents the transverse spatial coherence length.

For $\delta \to \infty$, one recovers the fully coherent case.

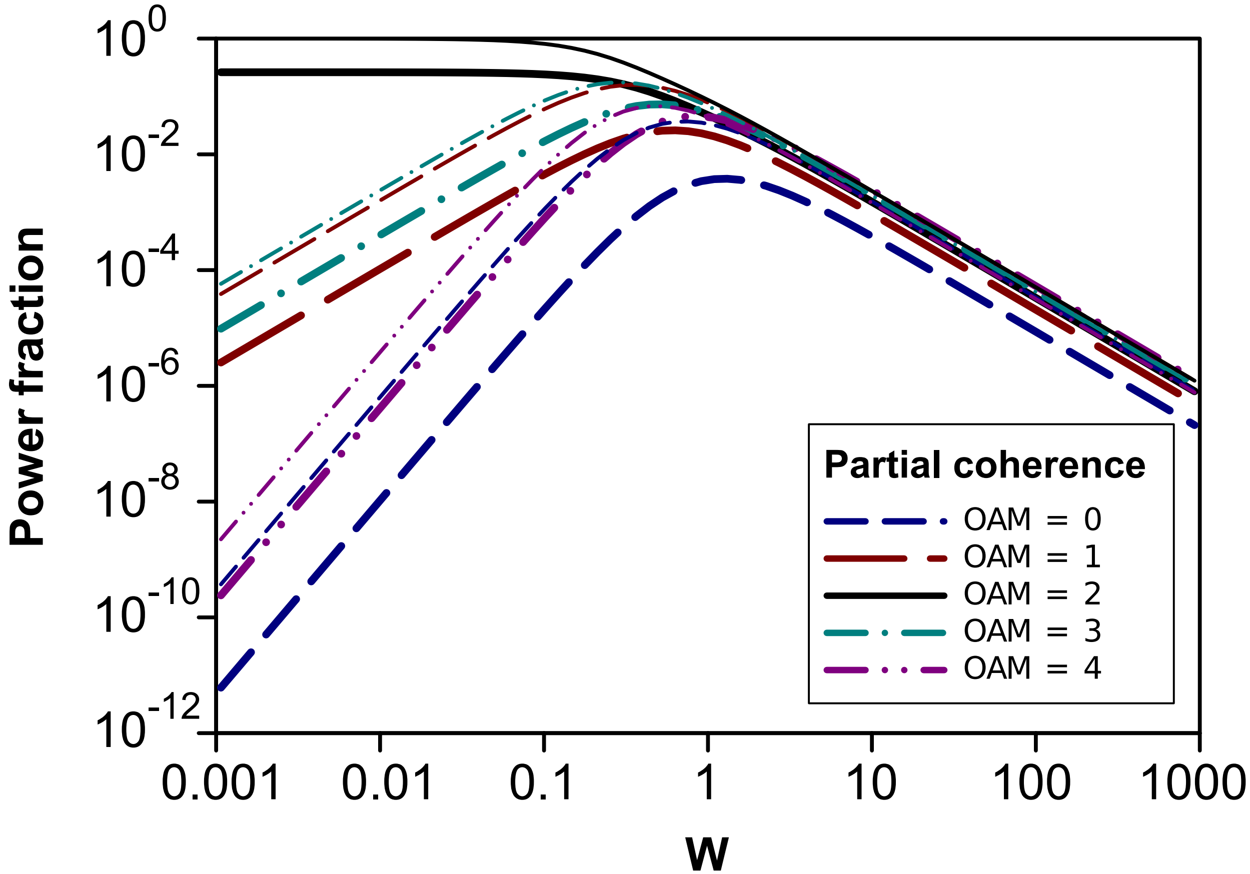
For $\delta = 3w_p$:



OAM scattering: output mode size

Use output mode size $w_{\rm out}$ (measurement basis)
that is different from the input mode size $w_{\rm in}$ (input basis)

For $w_{\rm out} = w_{\rm in}/2$:



Summary

- ▷ Turbulence distorts spatial modes $\Rightarrow$ loss of entanglement
- ▷ Investigate evolution of quantum states in turbulence
  - — Single phase screen (SPS) approach
  - — Multiple phase screen (MPS) approach
- ▷ Numerical simulations show where SPS breaks down
- ▷ Infinitesimal propagation equation (IPE) provides MPS approach
- ▷ Solve IPE without truncation in Fourier domain
- ▷ Application: OAM scattering in turbulence
- ▷ Allows us to investigate the effects of:
  - — weak turbulence
  - — partial coherence
  - — output mode size