Overlap Relation Between Free Space Laguerre Gaussian Modes and Step Index Fiber Modes

ROBERT BRÜNING[1], YINGWEN ZHANG[2], MELANIE MCLAREN[3], MICHAEL DUPARRÉ[1], AND ANDREW FORBES[3,*]

[1] Institute of Applied Optics, Abbe Center of Photonics, Friedrich Schiller University Jena, Fröbelstieg 1, D-07743 Jena, Germany
[2] Council for Scientific and Industrial Research, National Laser Center, P.O. Box 395, Pretoria 0001, South Africa
[3] School of Physics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa
[*] Corresponding author: andrew.forbes@wits.ac.za

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We investigated the overlap relation of the free space Laguerre Gaussian modes to the corresponding linearly polarized modes of a step index fiber. To maximize the overlap for an efficient coupling of the free space modes into a fiber, the scale dependent overlap was theoretically and experimentally determined. The presented studies paves the way for further improvement of free space to fiber optical connections.

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1. INTRODUCTION

Multi-mode fibers (MMF) are widely used in a plethora of applications such as optical sensors [1], fiber lasers [2] and optical communication [3]. Beside the possibility of all-fiber devices the connection between the fiber and its supported modes with the free space modes is important to control the optical properties of the emerging beam or the excited field distributions at the fiber input. The simplest connection is given by fibers with an parabolic refractive index profile, typical of graded-index fibers, whose modes can be described by Laguerre Gaussian functions and are also solutions of the free space wave equation [4]. However, in all other cases the fiber modes do not match the free space solutions, resulting in no stable propagation of the emerging beams in free space. Also, in general free space modes are not suitable to excite selective pure fiber modes and additional beam shaping techniques are required, such as the use of computer generated holograms [5]. The main disadvantages of such beam shaping techniques are the low efficiency and resulting high transformation losses. For the common type of step-index fibers, which show the same cylindrical symmetry as in free space, it is possible to approximate the fiber modes by suitable free space modes [6].

In this paper, we investigate the approximation of step-index fiber modes by free space modes theoretically as well as experimentally to evaluate the quality and the limitation of this approach. We determine the overlap relation between both mode sets as a function of the scale parameter for the free space as well as the fiber parameter, V. For the theoretical investigations we develop an analytical solution for this overlap problem, which allows us to study a wide parameter range. Additionally we investigate experimentally the overlap relation by applying the correlation filter method [7] and verify the analytical solution. Our results will be of interest to studies where fiber to free space links are necessary.

2. FUNDAMENTALS

At the transition from free space to fiber, and vice versa, one always has mode coupling between the free space modes on the one side and the fiber modes on the other. To achieve a maximized coupling efficiency and a low crosstalk between modes of different order, the scaling of the free-space beam has to be adapted to the fiber.

For the case of a weakly guiding step index fiber the fiber modes are given by the linearly polarized (LP) mode set. For that the field distribution $F_{lp}(r,\varphi)$ is given by the solution of the scalar Helmholtz-equation. Considering a cylindrically symmetric fiber with core radius a, the solution is given by [4]

$$LP_{lp}(r,\varphi) = N_{lp}\begin{cases} J_l(\frac{\nu_{lp} r}{a})/J_l(\nu_{lp}) & \text{for r<a} \\ K_l(\frac{\mu_{lp} r}{a})/K_l(\mu_{lp}) & \text{for r}\geq\text{ a}\end{cases} \times e^{il\varphi}, \quad (1)$$

where N_{lp} is a normalization constant, J_l denotes the lth order Bessel-function of the first kind and K_l denotes the lth order modified Bessel-function of the second kind, with ν_{lp} and μ_{lp} the normalized propagation constants of the core and cladding, respectively. The exact expression for the normalization constant is derived in the appendix. For the description of the LP modes additional the fiber parameter V, which is defined as

$$V^2 = \nu^2 + \mu^2 = \left(\frac{2\pi}{\lambda}a\right)^2 (n_{core}^2 - n_{cladding}^2) \tag{2}$$

is necessary to define the amount of modes and their propagation constants.
Since the step index fiber has cylindrical symmetry the adapted free space mode set is given by the Laguerre-Gaussian (LG) modes, which are the solutions of the paraxial Helmholtz-equation in a cylindrically symmetric coordinate system. For that the solution at the waist position is given by

$$LG_{pl}(r,\varphi) = M_{pl}\left(\frac{2r^2}{w_0^2}\right)^{\frac{|l|}{2}} L_p^{|l|}\left(\frac{2r^2}{w_0^2}\right) e^{-\frac{r^2}{w_0^2}} e^{il\varphi}, \tag{3}$$

where $L_p^{|l|}$ are the associated Laguerre polynomials and $M_{pl} = \frac{1}{w_0}\left(\frac{2p!}{\pi(l+p)!}\right)^{\frac{1}{2}}$ is a normalization factor.
Comparing the modes described by Eq. (1) and Eq. (3), we notice that they show the same azimuthal dependence and a characterization of the radial order by the amount of root points in the intensity distribution. Hence corresponding modes can be found by choosing the same azimuthal order and the field functions with the same amount of roots in radial direction.
Since both mode sets differ in the actual shape of the radial function and adaption of the scale parameters is needed for the best possible matching of both mode sets. The matching of the scale can be evaluated by the overlap relation

$$\eta_n = \iint LG_n(r,\varphi) LP_n^*(r,\varphi) dA, \tag{4}$$

where η_n defines the amount of power which is coupled from on mode into the over at the transition between both mode sets. It can reach values between one when the fields are perfectly matched and zero for orthogonal fields. The amount of power $1-\eta_n$ which is not coupled into the desired mode goes in other non-orthogonal, reflecting or radiating modes to satisfy energy conservation.

3. ANALYTICAL ANALYSIS

In order to optimize the mode overlap we derive an analytical express for the overlap relation as a function of the mode parameters. First we separate the problem into a core and a cladding part, corresponding to the solution of the fiber modes Eq. 1. The $e^{il\varphi}$ term insures that the LP and LG modes must have the same azimuthal index l. Hence to solve the overlap relation the radial part of Eq. 4 becomes the important one.
For deriving the solution for the core region we used that the Bessel function of the first kind can be written as a infinite sum

$$J_l(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+l+1)}\left(\frac{x}{2}\right)^{2m+l} \tag{5}$$

with Γ the gamma function, whereas the Laguerre polynomial can be written as it's generating function

$$L_p^{|l|}(x) = \frac{1}{p!}\left(\frac{d}{d\xi}\right)^p \frac{1}{(1-\xi)^{|l|+1}} exp\left(\frac{-x\xi}{1-\xi}\right)\Bigg|_{\xi=0}. \tag{6}$$

Using Eq. 5 and 6 for the description of the mode function Eq. 1 and Eq. 3 respectively the overlap of both modes set within the core is given by

$$\begin{aligned}\eta_{lp}^{core} =& \frac{N}{J_l(\nu)}\frac{M}{p!}\left(\sqrt{2}\right)^l \left(\frac{d}{d\xi}\right)^p \frac{1}{(1-\xi)^{|l|+1}} \\ &\times \sum_{m=0}^{\infty}\left\{\frac{(-1)^m}{m!\Gamma(m+l+1)}\left(\frac{\nu w}{2a}\right)^{2m+l}\left(\frac{1-\xi}{1+\xi}\right)^{m+l+1}\right. \\ &\times \left.\left[\Gamma(m+l+1) - \Gamma\left(m+l+1, \frac{a^2}{w^2}\frac{1+\xi}{1-\xi}\right)\right]\right\}\Bigg|_{\xi=0}, *\end{aligned} \tag{7}$$

with $\Gamma(a,z)$ the incomplete gamma function, N and M the normalization constant of the LP and LG mode respectively, a the fiber core radius and w the Gaussian radius.
For the cladding region a also a analytical representation for the overlap integral can be found by a similar approach. For that we make use of following relation between modified Bessel functions of the second kind K_l and the modified Bessel functions of first kind I_l

$$K_l(x) = \lim_{n\to l}\frac{\pi}{2\sin(n\pi)}\left[I_{-n}(x) - I_n(x)\right], \tag{8}$$

whereas the I_n can be written as an infinite sum

$$I_n(x) = \sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n}. \tag{9}$$

Using Eq. (8) and (9) to describe the field distribution of the fiber modes in the cladding region together with the previously used representation of the LG modes given by Eq. (3) and (6) the overlap of the cladding part becomes

$$\begin{aligned}\eta_{lp}^{cl} =& \lim_{n\to l}\frac{\pi}{2\sin(n\pi)}\frac{N}{K_n(\mu)}\frac{M}{p!}\left(\sqrt{2}\right)^n\left(\frac{d}{d\xi}\right)^p \\ &\times \frac{1}{(1-\xi)^{|l|+1}}\sum_{m=0}^{\infty}\left\{\frac{-1}{m!}\left(\frac{\mu w}{2a}\right)^{2m-l}\left(\frac{1-\xi}{1+\xi}\right)^{m+1}\right. \\ &\left[\frac{\Gamma\left(m+n+1, \frac{a^2}{w^2}\frac{1+\xi}{1-\xi}\right)}{\Gamma(m+n+1)}\left(\frac{\mu w}{2a}\right)^{2n}\left(\frac{1-\xi}{1+\xi}\right)^n\right. \\ &\left.\left.\frac{\Gamma\left(m+1, \frac{a^2}{w^2}\frac{1+\xi}{1-\xi}\right)}{\Gamma(m-n+1)}\right]\right\}\Bigg|_{\xi=0}.\end{aligned} \tag{10}$$

The complete overlap relation of core and cladding is than given by the sum of Eq. (7) and (10).

4. CORRELATION FILTER ANALYSIS

For a direct measurement of overlap relation between the LP and the corresponding LG modes the correlation filter method (CFM) was used. This performs all optically the integral relation given by Eq. 4 and allows the experimental investigation of the overlap between LP and LG modes [7].
The experimental setup used for the measuring of the scale depending overlap relation is shown in Fig. ??. A plane wave has illuminated a phase only spatial light modulator (SLM). Applying the phase only coding technique proposed by Arrizòn et al. [?] enables the shaping of arbitrarily scaled LG modes. The generated LG modes were then imaged by a telescopic 4f setup onto the CF, which had the field distribution of the LP modes implemented as the transmission function. After an optical Fourier transformation with a lens in 2f configuration the correlation signal was accessible at the CCD sensor, which was proportional to the overlap relation between both modes.

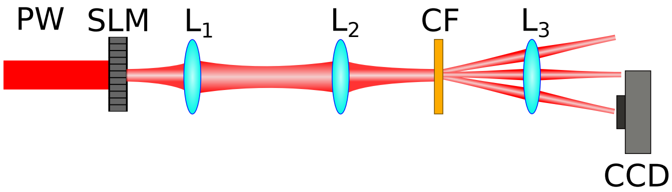

Fig. 1. The experimental setup for the measurement of the overlap relation between LP and LG modes, where the LG modes are generated with the SLM and decomposed by the CFM. PW - plane wave; SLM - spatial light modulator; L_{1-3} - lenses; CCD - camera

Since the encoding condition of the generated fields requires the normalization to unit amplitude the energy conservation is violated by scaling the LG mode size. To ensure the comparability of the results, a correction parameter for each encoded field was introduced, given by the maximal amplitude $\alpha_{\max}$ of the orthonormal field distribution. An appropriated power scaling and a correction of the measured intensities I_{mes} was achievable by subsequently applying of the correction coefficient for to a normalized intensity $I_{norm} = I_{mess}\alpha^2_{\max}$ [5]. As the last step the measured and corrected overlap values has to be normalized to one, which is given by the overlap between the LP mode and itself. For that the LP mode encoded in the correlation filter was generated with the SLM and the measured overlap relation was used to normalized the relations obtained for the different scaled LG modes.
This procedure for the generation of dynamic scaled LG modes and the evaluation of the overlap regarding a fixed scaled set of LP modes enabled the measurement of the scale depending overlap relation between both mode sets.

5. RESULTS

We applied the correlation filter method onto a LP mode set with a underlying V parameter of 4.72 resulting in the appearance of six guided modes. In Fig. 2 resulting scale depending overlap are depicted together with the theoretically curve calculated by our analytical solution. It can be seen that for all LP modes of the investigated example the corresponding LG modes are good approximation for a ratio between beam with and core radius of 0.75. In this example the overlap relations are about 0.99 for the LP_{01}, LP_{11} and, LP_{21} modes and about 0.98 for the LP_{02} mode. Comparing the scale depending overlap relations of the four fiber modes show explicit differences between them. Noticeable are the difference in the decrease of the overlap for not ideal scaled LG modes, whereas for the fundamental mode LP_{01} the overlap falls relatively slow, while the higher order modes show a obvious stronger decline. Additional the optimal ratio between beam width and core radius, for a maximized overlap, changes slightly for the investigated modes from 0.70 for the LP_{21}, see Fig. 2 (d), up to 0.81 for the fundamental mode LP_{01}, see Fig. 2 (a). Also is the maximal reachable overlap for the higher order radial LP_{02} mode, see Fig. 2 (b), with about 0.98 slightly lower compared to the other modes. Since the comparison between the measurement results and the theoretical curved achieved by the analytical solution showed a very high compliance and proof the reliability of our method, we used the theoretical values to study further the

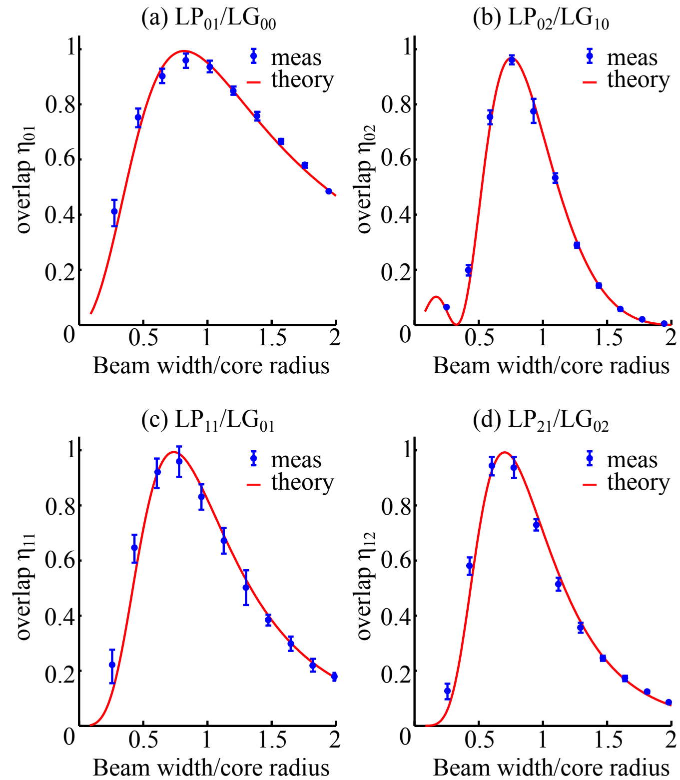

Fig. 2. Comparission between theoretical and experimental determined overlap relation

influences of the mode order or the V parameter on the overlap relation.

As seen in Fig. 2 the overlap relation change in shape, maximum position and maximum value for different modes. This means that there is always an optimal match, based on the fundamental Gaussian beam size, for the corresponding LG mode set. Applying our analytical model we have determined the scale depending overlap for different higher order mode for a fiber with $V = 50$ to demonstrate some effects. As shown in Fig. 3 for higher order radial modes with azimuthal index $l = 0$ and higher order azimuthal mode with radial index $p = 1$, the position of the best fitting beam size shifts to smaller Gaussian beam widths with increasing mode order. This behavior becomes intuitive since the beam width of LG modes of higher order follows $w = w_0\sqrt{2p + l + 1}$, with w_0 the Gaussian width. On the other side the LP modes are well confined within the core resulting in a decrease of the needed Gaussian width to match the scale of the corresponding modes. A second effect which can be seen is that the scale region with high overlap shrinks with increasing mode order and a good adaption of the scale parameters become more crucial. Especially for modes of higher radial order the maximal achievable overlap between both mode sets decrease with increasing mode order indicating the limits of the applied approximation. Hence the approximation of LP modes of the step index fiber by LG modes can not extend for arbitrary higher order modes and has to be considered in possible application with respect to the acceptable coupling losses introduced by this mismatch. Nevertheless this results demonstrate the ability to approximate lower order LP modes efficiently by LG modes.
Additionally we investigated the influence of the underlying V-parameter on the quality of our suggested approximation.

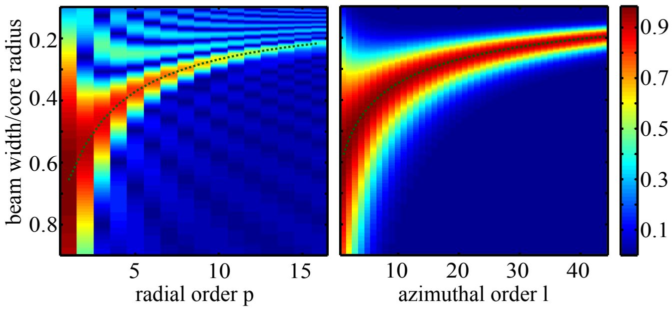

Fig. 3. Size depending overlap relation for modes with increasing radial and azimuthal order. The trend of the best fitting beams size is highlighted by the green line

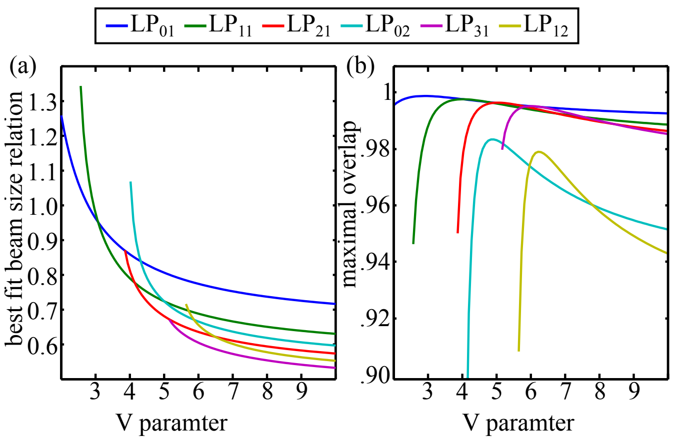

Fig. 4. The V parameter dependence of the overlap relation between LP corresponding LG modes. (a) The best relation between Gaussian beam width and core radius for a maximized overlap and (b) the maximal achievable overlap in dependence of V

For that we have calculated the best fitting beam size and the maximal achievable overlap for different modes as a function of the V parameter as shown in Fig. 4. In Fig. 4 (a) the change of the best fitting beam size is depicted. It can be seen that for low V values, in case of LP_{01}, or V values close to the cutoff of a mode the size relation becomes larger one indicating the bad confinement of the mode inside the core. For increasing V values the relation starts rapidly to decrease. A second effect occurs for week confined LP modes in fact that the overlap relation becomes relatively low and increases with increasing distance from the cut off up to some maximal value. Afterwards the highest achievable overlap starts to decrease, whereby the it drops faster for with increasing mode order.

Finally we have also compared the intensity profiles for the best fitting LG modes with the corresponding LP modes. Four examples for a fiber with a V parameter of 4.72 can be seen in Fig. 5, where the different mode profiles are normalized to unit intensity for better comparability. It can be clearly seen that the conformity decrease with increasing mode order. Figure 5 (a) illustrating the fundamental modes LP_{01} and LG_{00} respectively are nearly matched regarding their radial behavior wheres the distinctions become more significant for the higher order modes. For modes of higher azimuthal order like the LP_{11} and LG_{01} mode respectively, depicted in Fig. 5 (b), the maximums of ring like intensity pattern are slightly displaced. Also for

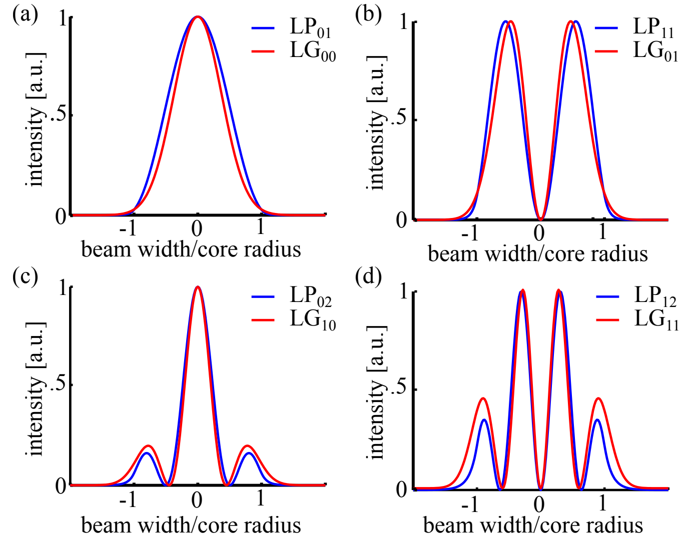

Fig. 5. Comparison of the intensity profiles between the LP modes and the corresponding LG modes for the maximized overlap.

higher order radial modes structural differences occur like the displacement of the roots and the side lobe maxima, see Fig. 5 (c). This distinctions become more relevant with increasing mode order as shown in Fig. 5 (d) for the higher order radial as well as azimuthal modes LP_{12} and LG_{21} respectively, where especially the side lobes show pregnant deviations resulting in a lower achievable overlap relation.

6. CONCLUSION

We have shown that the LP modes of a step-index fiber can be approximated by an scale adapted set of LG free space modes. The have proof the validity of this approximation experimentally for a example fiber with a V parameter of 4.72. Further we used a analytical solution of the overlap problem to investigate the limits of the approach for a wide spectrum of LP modes and V parameters. We have found that the approximation can used within certain limits. In order to provide high overlap the LP modes have to be far from there cutoff condition and of low radial order. Additionally the best fitting beam size change for each corresponding mode pair. For the few mode case like the example fiber, this dependence is week and a good approximation of all modes can be found for appropriated adapted LG mode set. In the case of highly multi mode fibers the used LG mode set has to be optimized individually for each corresponding mode pair or at least for different groups of them.

7. ACKNOWLEDGMENT

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8. APPENDIX

In order to evaluate the overlap relation between the LG and LP modes we need also a analytical expression for the normalization constant N_{lp} in Eq. (1). For that we use the normalization condition

$$1 = \iint LP_{lp}(r,\varphi)LP^{*}_{lp}(r,\varphi)dA \tag{11}$$

and gives

$$N^2_{lp} = \frac{1}{2\pi}\left\{\frac{\int_0^a \left[J_l(\frac{\nu_{lp}r}{a})\right]^2 rdr}{|J_l(\nu_{lp})|^2} + \frac{\int_a^\infty \left[K_l(\frac{\mu_{lp}r}{a})\right]^2 rdr}{|K_l(\mu_{lp})|^2}\right\}^{-1} \tag{12}$$

The integrals are evaluated to be

$$2\pi\int_0^a \left[J_l(\tfrac{\nu r}{a})\right]^2 rdr = \frac{\pi a^2}{\nu_{lp}}\left[\nu J_l(\nu)^2 - 2lJ_l(\nu)J_{l+1}(\nu) + \nu J_{l+1}(\nu)^2\right] \tag{13}$$

and

$$\begin{aligned} 2\pi\int_a^\infty \left[K_l(\tfrac{\mu r}{a})\right]^2 rdr = \lim_{n\to l}\frac{\pi^2a^2}{4\mu^2\sin(n\pi)^2}\Big\{&\pi\mu^2 I_{1-n}(\mu^2) \\ &+ 2\pi\mu I_{1-n}(\mu)\left[-\mu I_{n-1}(\mu) + nI_n(\mu)\right] \\ &- \pi\mu I_{-n}(\mu)^2 + \pi\left[2n\mu I_{n-1}(\mu)I_n(\mu)\right. \\ &\left.-(4n^2+\mu^2)I_n(\mu)^2 + \mu^2 I_{1+n}(\mu)^2\right] \\ &+ 2\mu I_{-n}(\mu)\left[2nK_{1-n}(\mu)\sin(n\pi)\right. \\ &\left.+\pi\mu I_n(\mu)\right]\Big\}. \end{aligned} \tag{14}$$

with $I_n(x)$ being the modified Bessel function of the first kind. Inserting Eq. (13) and Eq. (14) into Eq. (12) yields the analytical expression for the normalization constant for the solution of the overlap relation for the core and cladding region Eq. (7) and Eq. (10), respectively.

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