Lindblad equation for the decay of entanglement due to atmospheric scintillation

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Abstract. The quantum state for the spatial degrees of freedom of photons propagating through turbulence is analyzed. The turbulent medium is modeled by a single phase screen for weak scintillation conditions and by multiple phase screens for general scintillation conditions. In the former case the process is represented by an operator product expansion, leading to an integral expression that is consistent with current models. In the latter case the evolution of the density operator is described by a first order differential equation with respect to the propagation distance. It is shown that this differential equation is expressed in the Lindblad form, prior to the evaluation of the ensemble averages. After the evaluation of the ensemble averages the equation takes on the form of the infinitesimal propagation equation.

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1. Introduction

Studying how photonic states that are initially entangled in terms of their transverse spatial modes loose their entanglement as they propagate through atmospheric turbulence, one finds oneself within the overlap between two fields. One is the scintillation of classical optical beams, which has been studied for several decades [1, 2, 3, 4]. The other is the currently vibrant field of quantum information science [5] and in particular the study of open quantum systems [6] and quantum optics [7].

The problem of entanglement decay due to atmospheric scintillation has been considered by Paterson [8] from a conventional classical optics point of view, by assuming that one can represent the turbulent atmosphere as a single phase screen. This single phase screen model is currently used as the basis for most work that is being done on the decay of entangled photons propagating through atmospheric turbulence [9, 10, 11]. However, it is by construction only valid under weak scintillation conditions. Although the predictions of this approach have been shown to be consistent with experimental observations [12], one can rightfully ask whether this approach makes sense within the context of quantum information theory: does it represent a valid quantum process?

More recently, a multiple phase screen model has been proposed [13, 14]. However, it is also derived using a classical optics appoach and as such its validity as a quantum process also comes under question.

In this paper we consider the decay of the transverse spatial modal entanglement of photonic states propagating through turbulence from the perspective of quantum information theory. Within this context we consider both the single phase screen approach, which assumes weak scintillation, and the multiple phase screen approach without the assumption of weak scintillation. The single phase screen process, which one can express as an operator product expansion, leads to the same integral expression found from the classical optics derivation [8]. In the case of the multiple phase screen process, the resulting expression is in the form of a Lindblad equation [15, 16, 5, 6]. The latter indicates that the formalism represents a quantum process that produces a valid density operator. We also show that this result reproduces the equation in [13] after performing ensemble averaging.

The paper is organized as follows. First some basic aspects of turbulence and scintillation are reviewed in Sec. 2. In Sec. 3 we consider the single phase screen model from the perspective of a quantum process and in Sec. 4 we discuss the conditions for weak scintillation. In Sec. 5 we derive the Lindblad equation using the multiple phase screen approach. Finally we provide some conclusions in Sec. 6.

2. Turbulence and scintillation

Scintillation is the process of distortion that an optical beam experiences while propagating through a random medium such as atmospheric turbulence. A quantitative model for optical scintillation requires a model for the turbulence. There are a number of such models [1, 2, 3, 4] depending, among other things, on whether one includes the effects of the inner scale and outer scale. The simplest model is the Kolmogorov model, which is valid in the inertial range between the inner and outer turbulence scales. The Kolmogorov power spectral density of the refractive index fluctuations is given by [4]

$$\Phi_n(\mathbf{k}) = 0.033(2\pi)^3 C_n^2 |\mathbf{k}|^{-11/3}, \tag{1}$$

where C_n^2 is the refractive index structure constant. The strength of the turbulence is determined by C_n^2, with values ranging from about 10^{-17} m$^{-3/2}$ for weak turbulence to about 10^{-13} m$^{-3/2}$ for strong turbulence. The power spectral density is related to the autocorrelation function $B(\Delta\mathbf{r}) = \langle \tilde{n}(\mathbf{r}_1)\tilde{n}(\mathbf{r}_2)\rangle$, with $\Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, via the Wiener-Kintchine theorem

$$\Phi_n(\mathbf{k}) = \int B(\mathbf{r}) \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{r})\ \mathrm{d}^3 r. \tag{2}$$

The structure function, which is related to the autocorrelation function is defined by

$$D(\Delta\mathbf{r}) = \langle [\tilde{n}(\mathbf{r}_1) - \tilde{n}(\mathbf{r}_2)]^2 \rangle = 2\left[B(0) - B(\Delta\mathbf{r})\right], \tag{3}$$

and it can be obtained directly from interference measurements.

The strength of scintillation is not only determined by the stength of the turbulence, but also by the other relevant dimension parameters, namely the distance z over which the light propagates through turbulence and the wavelength of the light λ. These parameters are combined into the Rytov variance, given by

$$\sigma_R^2 = 1.23 C_n^2 k^{7/6} z^{11/6}, \tag{4}$$

where k is the wave number $(2\pi/\lambda)$. For plane waves, strong scintillation conditions is believed to exist when $\sigma_R^2 > 1$ and, for Gaussian beams (with radius ω_0), strong scintillation exists when [4]

$$\sigma_R^2 > \left(t + \frac{1}{t}\right)^{5/6}, \tag{5}$$

where $t = z/z_R$, with z_R being the Rayleigh range $(\pi\omega_0^2/\lambda)$.

3. Single phase screen model

Although it is not directly formulated in the language of quantum information theory, the single phase screen model [8] represents a valid quantum process — it can be expressed as an operator product expansion [5]. To show this, we proceed as follows. Under weak scintillation conditions, the turbulent atmosphere can be represented by a single phase modulation [8]. The corresponding quantum process is a single step process

$$\rho(z) = U\rho(0)U^{\dagger}, \tag{6}$$

where the unitary operator is represented by a single phase factor $U \sim \exp[\mathrm{i}\theta(x,y)]$.

Assuming that the original input density matrix is a pure state $\rho(0) = |\psi\rangle\langle\psi|$, one can express the individual output density matrix elements by

$$\rho_{mn}(z) = \langle m|U|\psi\rangle\langle\psi|U^{\dagger}|n\rangle, \tag{7}$$

where $|m\rangle$ represents a discrete orthogonal basis for the transverse modes, such as the Laguerre-Gaussian modes. We insert an identity operator, resolved in terms of the two-dimensional spatial coordinate basis, to get

$$\langle m|U|\psi\rangle = \int \langle m|\mathbf{r}\rangle\langle\mathbf{r}|U|\psi\rangle \, \mathrm{d}^2 r \tag{8}$$

$$\langle\psi|U^{\dagger}|n\rangle = \int \langle\psi|U^{\dagger}|\mathbf{r}\rangle\langle\mathbf{r}|n\rangle \, \mathrm{d}^2 r, \tag{9}$$

where $\mathbf{r}$ is the two-dimensional transverse position vector. The mode functions for the transverse spatial modes are given by $E_n(\mathbf{r}) = \langle\mathbf{r}|n\rangle$ and the single phase screen approximation leads to $\langle\mathbf{r}|U|\psi\rangle = \exp[\mathrm{i}\theta(\mathbf{r})]\psi(\mathbf{r})$, where $\psi(\mathbf{r}) = \langle\mathbf{r}|\psi\rangle$ is the input field and $\theta(\mathbf{r})$ is the phase function that is obtained from the refractive index fluctuations $\tilde{n}$ by an integration along the direction of propagation

$$\theta(x,y) = k\int_0^z \tilde{n}(x,y,z) \, dz. \tag{10}$$

The magnitude of the random phase modulation is therefore proportional to the propagation distance z.

The expression for the density matrix element in Eq. (7) then becomes

$$\rho_{mn}(z) = \iint E_m^*(\mathbf{r}_1)E_n(\mathbf{r}_2)\psi(\mathbf{r}_1)\psi^*(\mathbf{r}_2)\exp\left[\mathrm{i}\theta(\mathbf{r}_1) - \mathrm{i}\theta(\mathbf{r}_2)\right]\ \mathrm{d}^2r_1\ \mathrm{d}^2r_2. \tag{11}$$

The unitary operator U in Eq. (6) corresponds to a particular instance of the refractive index fluctuations. Since we do not presume to have complete knowledge of the medium at any particular time and therefore can only make statistical predictions about its effect, we need to compute the ensemble average over all refractive index fluctuations. The density matrix elements are therefore given by

$$\rho(z) = \left\langle U\rho(0)U^\dagger\right\rangle = \sum_s^N \frac{1}{N}\ U_s\rho(0)U_s^\dagger, \tag{12}$$

where the subscript s denotes a particular instance of the refractive index fluctuations. When the ensemble average is applied to the expression in Eq. (11), it only affects the exponential function containing the random phase modulations and leads to

$$\langle\exp\left[\mathrm{i}\theta(\mathbf{r}_1) - \mathrm{i}\theta(\mathbf{r}_2)\right]\rangle = \exp\left[-\frac{1}{2}D\left(|\mathbf{r}_1 - \mathbf{r}_2|\right)\right]. \tag{13}$$

Here $D(\cdot)$ is the phase structure function. For Kolmogorov turbulence, it is given by

$$D(x) = 6.88\left(\frac{x}{r_0}\right)^{5/3}, \tag{14}$$

in terms of the Fried parameter [17],

$$r_0 = 0.185\left(\frac{\lambda^2}{C_n^2 z}\right)^{3/5}. \tag{15}$$

The ensemble average of the density matrix element is therefore given by

$$\langle\rho_{mn}(z)\rangle = \iint E_m^*(\mathbf{r}_1)E_n(\mathbf{r}_2)\psi(\mathbf{r}_1)\psi^*(\mathbf{r}_1)\exp\left[\frac{-1}{2}D\left(|\Delta\mathbf{r}|\right)\right]\ \mathrm{d}^2r_1\ \mathrm{d}^2r_2. \tag{16}$$

One can turn the integration in Eq. (16) into a dimensionless integral by normalizing $\mathbf{r}_1$ and $\mathbf{r}_2$ by some dimension parameter, such as the beam radius ω_0. The resulting expression that follows from the integral thus only depends on the dimensionless combination ω_0/r_0. All the adjustable dimension parameters are contained in ω_0/r_0, including the propagation distance z. As a result, the complete z-dependence is determined by the ω_0/r_0-dependence inside the structure function.

The integral in Eq. (16) is not analytically tractable due to the power of 5/3 that appears in the structure function inside the argument of the exponential function. Often this problem is avoided by approximating the structure function in Eq. (14) by a quadratic structure function

$$D \sim \left(\frac{x}{r_0}\right)^{5/3} \rightarrow \left(\frac{x}{r_0}\right)^2. \tag{17}$$

Smith and Raymer [9] used the single phase screen model with the quadratic structure function approximation to calculate the concurrence [18] (a measure of qubit entanglement) as a function of ω_0/r_0 for a photon pair that is initially entangled as a

Bell state in terms of the azimuthal index of the Laguerre-Gauss modes. They found that the concurrence decays to zero when $\omega_0/r_0 \approx 1$. For larger values of the azimuthal index the concurrence survives up to larger values of ω_0/r_0. This has been confirmed by experimental measurements and numerical simulations [12].

4. Weak scintillation

Let's consider the conditions under which one can use the single phase screen approximation more carefully. It is stated that the single phase screen approximation assumes that the scintillation is weak. The strength of the scintillation is quantified by the Rytov variance σ_R^2, defined in Eq. (4). In Fig. 1, a diagram is used to represent the different regions in terms of the Rytov variance as a function of the normalized propagation distance t. Both axes are shown in logarithmic scales. For an optical beam propagating through turbulence with a particular strength of turbulence, the Rytov variance is proportional to $t^{11/6}$. In Fig. 1 three lines are shown that repesent the Rytov variance for optical beams propagating through turbulence with three different strengths. These lines start at the bottom of the diagram in the region of weak scintillation and move toward the region of strong scintillation at the top of the diagram. These lines cross the boundary between these regions, given by Eq. (5), more or less at the same point where they cross the line where $\omega_0/r_0 = 1$ (dashed line in Fig. 1). The dashed line is obtained from the Rytov variance by expressing it in terms of ω_0/r_0,

$$\sigma_R^2 = 1.637\ t^{5/6} \left(\frac{\omega_0}{r_0}\right)^{5/3} . \tag{18}$$

Hence, when $\omega_0/r_0 = 1$ the Rytov variance is proportional to $t^{5/6}$.

Since $\omega_0/r_0 = 1$ represents the approximate point where the concurrence goes to zero, the optical field never seems to arrive in the region of strong scintillation with non-zero entanglement, which seems to imply that the single phase screen model can be used for all situations. There are, however, some conditions under which the entanglement can survive into the region of strong scintillation, for instance, when larger values of azimuthal index are used. As a result, it does make sense to develop a reliable model that does not only apply in weak scintillation conditions.

5. Multiple phase screen approach

5.1. Operator product expansion

In the multiple phase screen approach the quantum process is broken up into multiple steps. Instead of doing the calculation in one step, going from the pure initial state to the final mixed state, as was done in Sec. 3, one can break the process up into infinitesimally small steps. In each step the infinitesimal propagation process operates on the density operator of a (potensially) mixed state and produces a slightly perturbed

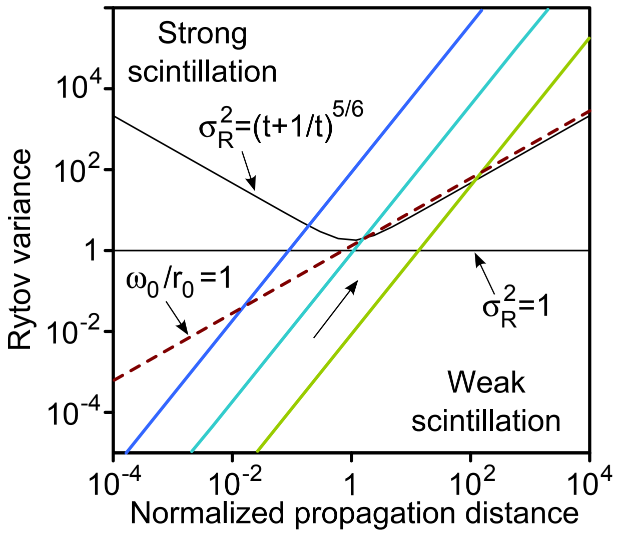

Figure 1. The different regions of scintillation strength are shown in terms of the Rytov variance σ_R^2 as a function of the normalized propagation distance t on a log-log scale. The boundaries between the region of weak scintillation at the bottom and the region of strong scintillation at the top, are shown for both plane waves ($\sigma_R^2 = 1$) and Gaussian beams [$\sigma_R^2 = (t + 1/t)^{5/6}$]. The dashed line represents the line where $\omega_0/r_0 = 1$. The three slanted coloured lines represent the scintillation strengths of optical beams propagating through different strengths of turbulence — from left to right: $C_n^2 = \{10^{-12}, 10^{-14}, 10^{-16}\}$ m$^{2/3}$.

version of this density operator

$$\rho(z) \rightarrow \rho(z + dz) = dU\rho(z)dU^{\dagger}. \tag{19}$$

Here dU represents the unitary operator for an infinitesimal propagation through turbulence. Again the ensemble averaging can be expression as a summation over the ensemble elements

$$\rho(z + dz) = \left\langle dU\rho(z)dU^{\dagger}\right\rangle = \sum_s^N \frac{1}{N} dU_s\rho(z)dU_s^{\dagger}, \tag{20}$$

where, as before, the subscript s denotes a particular instance of the refractive index fluctuations. In terms of the density matrix elements for a single photon, this becomes

$$\rho_{mn}(z + dz) = \sum_s^N \frac{1}{N} \sum_{pq} \langle m|dU_s|p\rangle \rho_{pq}(z) \langle q|dU_s^{\dagger}|n\rangle. \tag{21}$$

We'll first consider the density matrix for a single photon and then generalize it to the case with two photons.

5.2. Field transformation

Although the single phase screen approximation is valid in this situation, thanks to the short propagation distances, the fact that this operation is performed repeatedly implies that one also needs to include the free space propagation process, which is neglected in the single phase screen model.‡ As a result dU will contain a propagation term in addition to the random phase modulation term. To find the correct expression for dU we start with the equation of motion in turbulence, which is given by [4]

$$\nabla_T^2 g(\mathbf{x}) - \mathrm{i}2k\partial_z g(\mathbf{x}) + 2k^2\tilde{n}(\mathbf{x})g(\mathbf{x}) = 0. \tag{22}$$

Here $g(\mathbf{x})$ represents the scalar electromagnetic field and $\tilde{n}(\mathbf{x})$ is the fluctuations in the refractive index. The expression in Eq. (22) is obtained from the Helmholtz equation under the paraxial approximation, which assumes that the optical field propagates close to the beam axis (in this case the z-axis) and under the assumption that $\tilde{n} \ll \langle n\rangle \approx 1$. The conditions for both these approximations are well satisfied in our situation.

In the two-dimensional transverse Fourier domain Eq. (22) becomes

$$-4\pi^2|\mathbf{a}|^2 G(\mathbf{a},z) - \mathrm{i}2k\partial_z G(\mathbf{a},z) + 2k^2 N(\mathbf{a},z)\star G(\mathbf{a},z) = 0, \tag{23}$$

where $\mathbf{a}$ is the two-dimensional spatial frequency vector (related to the two-dimensional momentum vector by $\mathbf{K} = 2\pi\mathbf{a}$), $\star$ denotes convolution and $G(\mathbf{a},z)$ and $N(\mathbf{a},z)$ are the two-dimensional transverse Fourier transforms of $g(\mathbf{x})$ and $\tilde{n}(\mathbf{x})$, given by

$$G(\mathbf{a},z) = \int g(\mathbf{x}) \exp[\mathrm{i}2\pi(ax+by)]\ \mathrm{d}x\mathrm{d}y \tag{24}$$

and

$$N(\mathbf{a},z) = \int \tilde{n}(\mathbf{x}) \exp[\mathrm{i}2\pi(ax+by)]\ \mathrm{d}x\mathrm{d}y, \tag{25}$$

respectively. It then follows that

$$G(\mathbf{a},z_0+dz) = G(\mathbf{a},z_0) + \frac{\mathrm{i}dz}{2k}\left[4\pi^2|\mathbf{a}|^2 G(\mathbf{a},z_0) - 2k^2 N(\mathbf{a},z0)\star G(\mathbf{a},z_0)\right]. \tag{26}$$

One can use $G(\mathbf{a},z)$ to define a quantum state in terms of a two-dimensional momentum basis. For instance,

$$|m\rangle = \int |\mathbf{a}\rangle G_m(\mathbf{a},z)\ \mathrm{d}^2a, \tag{27}$$

where $|\mathbf{a}\rangle$ ($\equiv$ $|\mathbf{K}\rangle$) denotes the two-dimensional momentum basis elements and $G_m(\mathbf{a},z) = \langle\mathbf{a}|m\rangle$. Note that if we substitute $G(\mathbf{a},z_0) = G_m(\mathbf{a},z_0)$ in Eq. (26), then one cannot assume that the transformed wave function is still associated with the same basis element $G(\mathbf{a},z_0+dz) \neq G_m(\mathbf{a},z_0+dz)$. This is due to the distortion introduced by the noise term that contains $N(\mathbf{a},z_0)$.

Finally, we use Eqs. (26) and (27) to express the quantum operation for the infinitesimal propagation through turbulence as

$$\begin{aligned}\langle m|dU_s|p\rangle = \delta_{mp} + \frac{\mathrm{i}dz}{2k}\int G_m^*(\mathbf{a},z_0)\left[4\pi^2|\mathbf{a}|^2 G_p(\mathbf{a},z_0)\right.\\ \left.-2k^2 N_s(\mathbf{a},z_0)\star G_p(\mathbf{a},z_0)\right]\ \mathrm{d}^2a\\ = \delta_{mp} + \mathrm{i}dz\ \mathcal{P}_{mp} + dz\ \mathcal{L}_{s,mp},\end{aligned} \tag{28}$$

‡ One can neglect the propagation process in the single phase screen model, because the inner products before and after a single propagation will give the same result.

where, in the last line, we defined

$$\mathcal{P}_{mp}(z) = \frac{2\pi^2}{k} \int |\mathbf{a}|^2 G_m^*(\mathbf{a}, z) G_p(\mathbf{a}, z)\ \mathrm{d}^2 a \tag{29}$$

and

$$\mathcal{L}_{s,mp}(z) = -\mathrm{i}k \iint G_m^*(\mathbf{a}, z) N_s(\mathbf{a} - \mathbf{a}', z) G_p(\mathbf{a}', z)\ \mathrm{d}^2 a\ \mathrm{d}^2 a'. \tag{30}$$

From Eqs. (29) and (30), we respectively note that $\mathcal{P}_{mp}$ is hermitian and that $\mathcal{L}_{s,mp}$ is anti-hermitian,

$$\mathcal{P}_{mp}^\dagger = \mathcal{P}_{pm}^* = \mathcal{P}_{mp} \tag{31}$$

$$\mathcal{L}_{s,mp}^\dagger = \mathcal{L}_{s,pm}^* = -\mathcal{L}_{s,mp}, \tag{32}$$

following from fact that $N_s^*(\mathbf{a}, z_0) = N_s(-\mathbf{a}, z_0)$, because $\tilde{n}(\mathbf{x})$ is a real-valued function.

The hermitian adjoint operator for the infinitesimal propagation through turbulence is then given by

$$\langle q|dU_s^\dagger|n\rangle = \delta_{qn} - \mathrm{i}dz\ \mathcal{P}_{qn}^\dagger + dz\ \mathcal{L}_{s,qn}^\dagger. \tag{33}$$

5.3. Hamiltonian

For a Hamiltonian based approach, one can determine the required 'Hamiltonian' directly from the definition of the unitary operator in Eq. (28), by assuming that z can be treated as 'time' and that we work in units where $\hbar = 1$. This Hamiltonian is given by

$$\mathcal{H}_{mp} = \mathcal{P}_{mp} - \mathrm{i}\mathcal{L}_{s,mp}, \tag{34}$$

where $\mathcal{P}_{mp}$ is interpreted as the kinetic term and $\mathcal{L}_{s,mp}$ as the potential or 'interaction' term. In the present case, however, (and this is very important) the field with which the optical field g 'interacts' is not a quantum field, but rather a classical field that represents the fluctuating refractive index $\tilde{n}$. Moreover, $\tilde{n}$ is not a dynamical field — it does not have a kinetic term. As a result the potential is equivalent to a mass term and the Hamiltonian is completely linear. If $\tilde{n}$ were a dynamical quantum field, a consistent formulation of this process would have required quantum field theory [19]. A consequence of this observation is that, when the Markov approximation is introduced in the derivation of the Lindblad equation it differs from the Markov approximation usually associated with a system coupled to a thermal bath. Here there is no coupling and the thermal bath is replaced by the classical refractive index fluctuations. As a result the Markov approximation used here is equivalent to that which is used in classical scintillation theory.

5.4. Second order in N_s

Substituting Eqs. (28) and (33) into Eq. (21), and expanding the result to first order in dz, we obtain

$$\rho_{mn}(z_0 + dz) = \rho_{mn}(z_0) + \mathrm{i}dz\,[\mathcal{P}, \rho(z_0)]_{mn}$$

$$+ dz \sum_{s}^{N} \frac{1}{N} \left[\mathcal{L}_{s,mp}(z_0)\rho_{pn}(z_0) + \rho_{mq}(z_0)\mathcal{L}_{s,qn}^{\dagger}(z_0) \right] \quad (35)$$

where we used Eq. (31) to express the kinetic part as a commutator. We also use the summation convention for all indices (except s), which means that there is an implied summation for all repeated indices. In this form the dissipative term [last term in Eq. (35)] would vanish if one performs ensemble averaging, because the ensemble average of refractive index fluctuations is zero $\langle N_s \rangle = 0$. One needs an expression that is second order in N_s to obtain nonzero dissipative terms.

Considering the equation in Eq. (35) for one instance of the refractive index, we turn it into a first order differential equation with respect to z

$$\partial_z \rho_{mn}(z) = \mathrm{i} \left[\mathcal{P}, \rho(z) \right]_{mn} + \mathcal{L}_{s,mp}(z)\rho_{pn}(z) + \rho_{mq}(z)\mathcal{L}_{s,qn}^{\dagger}(z). \quad (36)$$

Next we integrate Eq. (36) from z_0 to z to obtain

$$\rho_{mn}(z) = \rho_{mn}(z_0) + \mathrm{i} \int_{z_0}^{z} \left[\mathcal{P}, \rho(z_1) \right]_{mn} \, \mathrm{d}z_1 + \int_{z_0}^{z} \left[\mathcal{L}_{s,mp}(z_1)\rho_{pn}(z_1) + \rho_{mq}(z_1)\mathcal{L}_{s,qn}^{\dagger}(z_1) \right] \, \mathrm{d}z_1. \quad (37)$$

Then we substitute Eq. (37) back into itself repeatedly until we have second order terms in N_s containing density matrix elements that only depend on z_0.

The expressions for $\mathcal{P}$ and $\mathcal{L}$ given in Eqs. (29) and (30) contain the momentum space wave functions G_m and G_m^*, evaluated at z. However, under the paraxial approximation all modal functions are slow varying in z. As a result, for short enough distances, $G_m(\mathbf{a}, z) \approx G_m(\mathbf{a}, z_0)$. Hence, one can replace all these functions with their equivalents evaluated at z_0.

One can then evaluate the z integral for the kinetic term and discard terms beyond the first order in $(z - z_0)$. (Eventually we'll set $z - z_0 = dz$.) We also discard terms that are first order in N_s, because $\langle N_s \rangle = 0$. The resulting equation is

$$\begin{aligned} \rho_{mn}(z) = \; & \rho_{mn}(z_0) + \mathrm{i}(z - z_0) \left[\mathcal{P}, \rho(z_0) \right]_{mn} \\ & + \int_{z_0}^{z} \int_{z_0}^{z_1} \left[-\mathcal{L}_{s,mp}^{\dagger}(z_1)\mathcal{L}_{s,pq}(z_2)\rho_{qn}(z_0) + \mathcal{L}_{s,mp}(z_1)\rho_{pq}(z_0)\mathcal{L}_{s,qn}^{\dagger}(z_2) \right. \\ & \left. + \mathcal{L}_{s,mp}(z_2)\rho_{pq}(z_0)\mathcal{L}_{s,qn}^{\dagger}(z_1) - \rho_{mp}(z_0)\mathcal{L}_{s,pq}^{\dagger}(z_2)\mathcal{L}_{s,qn}(z_1) \right] \mathrm{d}z_2 \, \mathrm{d}z_1, \end{aligned} \quad (38)$$

where we used Eq. (32) to change some of the $\mathcal{L}$'s into $\mathcal{L}^{\dagger}$'s and visa versa. Note that the two z-integrations in the dissipative term are coupled. Moreover, the result after the two integrations over z must be linear in $(z - z_0)$ in order to give a first order differential equation in z.

The kinetic term in Eq. (38) already has the correct form as required for the Lindblad equation. The sign change is a result of the fact that the evolution is in terms of space and not time. One needs to remember that the 'Hamiltonian' in this case is actually a momentum operator. For this reason we have chosen a $\mathcal{P}$, instead of $\mathcal{H}$, to represent its free part.

5.5. Markov approximation

The dissipative term in Eq. (38) needs more careful attention. After replacing all momentum space wave functions in Eq. (30) with their equivalents evaluated at z_0, the only remaining z_1- and z_2-dependences are associated with the N_ss. Moreover, the ensemble averaging only operates on the products of N_ss.

The dissipative term, therefore, contains integrals of the form

$$\Gamma_0(\mathbf{a}_1,\mathbf{a}_2) = \int_{z_0}^{z}\int_{z_0}^{z_1} \langle N_s(\mathbf{a}_1,z_1)N_s^*(\mathbf{a}_2,z_2)\rangle\ \mathrm{d}z_2\ \mathrm{d}z_1. \tag{39}$$

One way to define N_s, which is consistent with Eqs. (2) and (3), is

$$N_s(\mathbf{a},z) = \int \exp(-\mathrm{i}k_z z)\tilde{\chi}_s(\mathbf{k}) \left[\frac{\Phi_0(\mathbf{k})}{\Delta^3}\right]^{1/2} \frac{\mathrm{d}k_z}{2\pi}, \tag{40}$$

where $\tilde{\chi}_s(\mathbf{k})$ is a three-dimensional, frequency domain, normally distributed, random complex function and Δ is an associated correlation length in the frequency domain. The fact that $\tilde{n}$ is a real-valued function implies that $\tilde{\chi}^*(\mathbf{k}) = \tilde{\chi}(-\mathbf{k})$. Furthermore, it is assume that this random function is delta-correlated,

$$\langle\tilde{\chi}(\mathbf{k}_1)\tilde{\chi}^*(\mathbf{k}_2)\rangle = (2\pi\Delta)^3\, \delta_3(\mathbf{k}_1-\mathbf{k}_2). \tag{41}$$

We now substituting Eq. (40) into Eq. (39) and evaluate the ensemble average with the aid of Eq. (41). Then we evaluate one of the k_z-integrals to obtain

$$\Gamma_0(\mathbf{a}_1,\mathbf{a}_2) = \delta_2(\mathbf{a}_1-\mathbf{a}_2) \int\int_{z_0}^{z}\int_{z_0}^{z_1} \Phi_0(\mathbf{k}_1) \times \exp[\mathrm{i}k_{1z}(z_2-z_1)]\ \mathrm{d}z_2\ \mathrm{d}z_1\ \frac{\mathrm{d}k_{1z}}{2\pi}. \tag{42}$$

Evaluating the two z-integrals and dropping terms that are anti-symmetric in k_{1z}, we obtain

$$\Gamma_0(\mathbf{a}_1,\mathbf{a}_2) = \delta_2(\mathbf{a}_1-\mathbf{a}_2) \int \Phi_0(\mathbf{k}_1) \frac{1-\cos[k_{1z}(z-z_0)]}{k_{1z}^2}\ \frac{\mathrm{d}k_{1z}}{2\pi}. \tag{43}$$

The z-dependent function is sharply peaked at $k_{1z}=0$ provided that $(z-z_0)$ is larger than the correlation length of the refractive index fluctuations. Under these conditions one can substitute $k_{1z}=0$ in $\Phi_0(\mathbf{k}_1)$ and evaluate the remaining integral over k_{1z}. The latter condition represents the Markov approximation, as used in classical scintillation theory [4]. Here it is applicable because $\tilde{n}$ is a classical field. The resulting expression is

$$\Gamma_0(\mathbf{a}_1,\mathbf{a}_2) = \frac{dz}{2}\delta_2(\mathbf{a}_1-\mathbf{a}_2)\Phi_1(\mathbf{a}_1), \tag{44}$$

where we substituted $z=z_0+dz$ and defined $\Phi_1(\mathbf{a}_1) = \Phi_0(2\pi\mathbf{a}_1, 0)$. Note that, although we had two z-integrals, we ended up with only one factor of dz.

5.6. Lindblad form

One can obtain the same expression for $\Gamma_0(\mathbf{a}_1, \mathbf{a}_2)$ as given in Eq. (44), by using

$$\Gamma_0(\mathbf{a}_1, \mathbf{a}_2) = \frac{dz}{2}\langle M_s(\mathbf{a}_1) M_s^*(\mathbf{a}_2)\rangle, \tag{45}$$

instead of Eq. (39), where

$$M_s(\mathbf{a}) = \tilde{\xi}_s(\mathbf{a}) \frac{[\Phi_1(\mathbf{a})]^{1/2}}{\Delta}, \tag{46}$$

with $\tilde{\xi}_s(\mathbf{a})$ being a two-dimensional, frequency domain, normally distributed, random complex function, such that $\tilde{\xi}^*(\mathbf{a}) = \tilde{\xi}(-\mathbf{a})$ and

$$\langle \tilde{\xi}(\mathbf{a}_1) \tilde{\xi}^*(\mathbf{a}_2)\rangle = \Delta^2 \delta_2(\mathbf{a}_1 - \mathbf{a}_2). \tag{47}$$

In effect, the z-integration is implicit in M_s. Hence, unlike N_s, M_s is independent of z.

One can remove the z-integrals in Eq. (38) by introducing a factor of $dz/2$ and replacing the $\mathcal{L}$-operators by new L-operators (Lindblad operators), defined by

$$L_{s,mp}(z_0) = -\mathrm{i}k \iint G_m^*(\mathbf{a}, z_0) M_s(\mathbf{a} - \mathbf{a}') G_p(\mathbf{a}', z_0)\ \mathrm{d}^2 a\ \mathrm{d}^2 a'. \tag{48}$$

The expression in Eq. (38) then becomes

$$\begin{aligned}\rho_{mn}(z_0 + dz) = \rho_{mn}(z_0) + \mathrm{i}dz\,[\mathcal{P}, \rho(z_0)]_{mn} + \frac{dz}{2}\big[2L_{s,mp}(z_0)\rho_{pq}(z_0)L_{s,qn}^\dagger(z_0) \\ -L_{s,mp}^\dagger(z_0)L_{s,pq}(z_0)\rho_{qn}(z_0) - \rho_{mp}(z_0)L_{s,pq}^\dagger(z_0)L_{s,qn}(z_0)\big],\end{aligned} \tag{49}$$

where we substituted $z = z_0 + dz$. One can now introduce the ensemble averaging, as expressed by the summation over all instances of the refracive index fluctuations, and convert the result into a first order differetial equation in z. The resulting expression is equivalent to a master equation in the Lindblad form

$$\begin{aligned}\partial_z\rho_{mn}(z) = \mathrm{i}\,[\mathcal{P}, \rho(z)]_{mn} + \frac{1}{2}\sum_s^N \frac{1}{N}\big[2L_{s,mp}(z)\rho_{pq}(z)L_{s,qn}^\dagger(z) \\ -L_{s,mp}^\dagger(z)L_{s,pq}(z)\rho_{qn}(z) - \rho_{mp}(z)L_{s,pq}^\dagger(z)L_{s,qn}(z)\big].\end{aligned} \tag{50}$$

The redefinition of the noise spectra N_s in terms of purely two-dimensional functions M_s is not necessary for the calculation of the equation, but it allows one to show that the equation can be expressed as a Lindblad equation. As a result one can conclude that the density matrix that solves this equation obeys the requirements of unity trace and positivity.

We now proceed to evaluate the ensemble averages in the expression. This can be done either in terms of N_s or M_s, since both give the same result. For convenience we proceed with M_s

5.7. Ensemble averaging

The three ensemble averages in Eq. (50) can all be obtained from

$$\begin{aligned}\Lambda_{mnpq} &= \sum_s^N \frac{1}{N} L_{s,mp}(z) L_{s,qn}^{\dagger}(z) = \langle L_{s,mp}(z) L_{s,qn}^{\dagger}(z)\rangle \\ &= k^2 \iiiint G_m^*(\mathbf{a}_1, z) G_p(\mathbf{a}_2, z) G_q^*(\mathbf{a}_3, z) G_n(\mathbf{a}_4, z) \\ &\quad \times \langle M_s(\mathbf{a}_1 - \mathbf{a}_2) M_s(\mathbf{a}_3 - \mathbf{a}_4)\rangle \ \mathrm{d}^2a_1\ \mathrm{d}^2a_2\ \mathrm{d}^2a_3\ \mathrm{d}^2a_4. \end{aligned} \tag{51}$$

Redefining $\mathbf{a}_1 \to \mathbf{a}_1 + \mathbf{a}_2$ and $\mathbf{a}_4 \to \mathbf{a}_3 + \mathbf{a}_4$, we have

$$\begin{aligned}\Lambda_{mnpq} = k^2 \iiiint & G_m^*(\mathbf{a}_1 + \mathbf{a}_2, z) G_p(\mathbf{a}_2, z) G_q^*(\mathbf{a}_3, z) G_n(\mathbf{a}_3 + \mathbf{a}_4, z) \\ & \times \langle M_s(\mathbf{a}_1) M_s^*(\mathbf{a}_4)\rangle \ \mathrm{d}^2a_1\ \mathrm{d}^2a_2\ \mathrm{d}^2a_3\ \mathrm{d}^2a_4. \end{aligned} \tag{52}$$

The ensemble average over the products of M_ss gives

$$\langle M_s(\mathbf{a}_1) M_s^*(\mathbf{a}_4)\rangle = \langle M_s(\mathbf{a}_1) M_s(-\mathbf{a}_4)\rangle = \Phi_1(\mathbf{a}_1)\delta_2(\mathbf{a}_1 - \mathbf{a}_4). \tag{53}$$

We substitute Eq. (53) into Eq. (52) and evaluate the integral over $\mathbf{a}_4$, to obtain

$$\begin{aligned}\Lambda_{mnpq} &= k^2 \iiint G_m^*(\mathbf{a}_1 + \mathbf{a}_2, z) G_p(\mathbf{a}_2, z) G_q^*(\mathbf{a}_3, z) G_n(\mathbf{a}_3 + \mathbf{a}_1, z) \\ &\quad \times \Phi_1(\mathbf{a}_1)\ \mathrm{d}^2a_1\ \mathrm{d}^2a_2\ \mathrm{d}^2a_3 \\ &= k^2 \int W_{mp}^*(\mathbf{a}, z) W_{nq}(\mathbf{a}, z) \Phi_1(\mathbf{a})\ \mathrm{d}^2a \end{aligned} \tag{54}$$

where

$$W_{ab}(\mathbf{a}, z) = \int G_a(\mathbf{a}' + \mathbf{a}, z) G_b^*(\mathbf{a}', z)\ \mathrm{d}^2a'. \tag{55}$$

Next we consider the case where two of the indices are contracted

$$\begin{aligned}\sum_p \Lambda_{mnpp} = k^2 \sum_p \iiint & G_m^*(\mathbf{a}_1 + \mathbf{a}_2, z) G_p(\mathbf{a}_2, z) G_p^*(\mathbf{a}_3, z) \\ & \times G_n(\mathbf{a}_3 + \mathbf{a}_1, z) \Phi_1(\mathbf{a}_1)\ \mathrm{d}^2a_1\ \mathrm{d}^2a_2\ \mathrm{d}^2a_3. \end{aligned} \tag{56}$$

The contraction leads to the completeness condition for these basis functions

$$\sum_p G_p(\mathbf{a}_2, z) G_p^*(\mathbf{a}_3, z) = \delta_2(\mathbf{a}_2 - \mathbf{a}_3). \tag{57}$$

Note that this completeness condition only applies in the infinite dimensional case where all basis functions are considered. Hence, the corresponding density matrix is infinite dimensional. Substituting Eq. (57) into Eq. (56) and evaluating the integral over $\mathbf{a}_3$, we obtain

$$\begin{aligned}\sum_p \Lambda_{mnpp} &= k^2 \iiint G_m^*(\mathbf{a}_1 + \mathbf{a}_2, z)\delta_2(\mathbf{a}_2 - \mathbf{a}_3) G_n(\mathbf{a}_3 + \mathbf{a}_1, z) \\ &\qquad \times \Phi_1(\mathbf{a}_1)\mathrm{d}^2a_1\ \mathrm{d}^2a_2\ \mathrm{d}^2a_3 \\ &= k^2 \iint G_m^*(\mathbf{a}_1 + \mathbf{a}_2, z) G_m(\mathbf{a}_2 + \mathbf{a}_1, z) \\ &\qquad \times \Phi_1(\mathbf{a}_1)\ \mathrm{d}^2a_1\ \mathrm{d}^2a_2. \end{aligned} \tag{58}$$

Next we redefine $\mathbf{a}_2 \rightarrow \mathbf{a}_2 - \mathbf{a}_1$. The resulting integral over $\mathbf{a}_2$ is an orthogonality relationship that gives a Kronecker delta

$$\int G_m^*(\mathbf{a}_2, z) G_n(\mathbf{a}_2, z)\ \mathrm{d}^2 a_2 = \delta_{mn}. \tag{59}$$

After we substitute Eq. (59) into Eq. (58), the resulting expression becomes

$$\sum_p \Lambda_{mnpp} = \delta_{mn} k^2 \int \Phi_1(\mathbf{a})\ \mathrm{d}^2 a = \delta_{mn} \Lambda_T. \tag{60}$$

5.8. Infinitesimal propagation equation

We now use the expressions and definitions in Eqs. (51-60) to simplify Eq. (50)

$$\partial_z \rho_{mn}(z) = i\mathcal{P}_{mp}(z)\rho_{pn}(z) - i\rho_{mp}(z)\mathcal{P}_{pn}(z) + \Lambda_{mnpq}(z)\rho_{pq}(z) - \Lambda_T \rho_{mn}(z). \tag{61}$$

The expression in Eq. (61) is the infinitesimal propagation equation (IPE) for a single photon propagating through turbulence.

5.9. Generalizations

We now generalize the single photon expression in Eq. (61) for two photon (bi-partite) states. The density operator becomes a density tensor contracted with the bra- and ket-basis vectors for both photons

$$\rho = \sum_{m,n} |m\rangle|p\rangle\ \rho_{mnpq}\ \langle n|\langle q|. \tag{62}$$

Here $|m\rangle$ and $\langle n|$ are, respectively, the ket- and bra-basis vectors for the one photon, and $|p\rangle$ and $\langle q|$ are, respectively, the ket- and bra-basis vectors for the other photon.

First we consider the case where only one of the two photons propagates through turbulence. The other photon is propagates through free-space without turbulence. The IPE for this case contains the free-space kinetic terms for both photons, but only dissipative terms form one of the photons. The resulting expression is thus given by

$$\begin{aligned} \partial_z \rho_{mnpq} = {} & i\mathcal{P}_{mx}\rho_{xnpq} - i\rho_{mxpq}\mathcal{P}_{xn} + i\mathcal{P}_{px}\rho_{mnxq} - i\rho_{mnpx}\mathcal{P}_{xq} \\ & + \Lambda_{mnxy}\rho_{xypq} - \Lambda_T \rho_{mnpq}. \end{aligned} \tag{63}$$

Next we consider the case where both photons propagate through turbulence, but along different paths, so that the turbulent media are uncorrelated. As a result the expression now contains dissipative terms for both photons

$$\begin{aligned} \partial_z \rho_{mnpq} = {} & i\mathcal{P}_{mx}\rho_{xnpq} - i\rho_{mxpq}\mathcal{P}_{xn} + i\mathcal{P}_{px}\rho_{mnxq} - i\rho_{mnpx}\mathcal{P}_{xq} \\ & + \Lambda_{mnxy}\rho_{xypq} + \Lambda_{pqxy}\rho_{mnxy} - 2\Lambda_T \rho_{mnpq}. \end{aligned} \tag{64}$$

This expression corresponds to the one obtained in [13].

Finally we consider a case where both photons propagate through the same turbulent medium, which allows for additional correlation terms to appear [20]

$$\begin{aligned} \partial_z \rho_{mnpq} = {} & i\mathcal{P}_{mx}\rho_{xnpq} - i\rho_{mxpq}\mathcal{P}_{xn} + i\mathcal{P}_{px}\rho_{mnxq} - i\rho_{mnpx}\mathcal{P}_{xq} \\ & + \Lambda_{mnxy}\rho_{xypq} + \Lambda_{pqxy}\rho_{mnxy} + \Lambda_{mqxy}\rho_{xnpy} + \Lambda_{pnxy}\rho_{myxq} \\ & - \Lambda_{xnqy}\rho_{mypx} - \Lambda_{mxyp}\rho_{ynxq} - 2\Lambda_T \rho_{mnpq}. \end{aligned} \tag{65}$$

6. Conclusions

A master equation in the Lindblad is derived for the process of entanglement decay in the spatial degrees of freedom that a photonic state experiences while propagating through turbulence. The derivation follows a multiple phase screen approach, which overcomes the requirement for weak scintillation conditions, as in the case of the single phase screen model. The Lindblad equation is obtained prior to the evaluation of the ensemble averages over all possible refractive index fluctuations. Once the ensemble averages are evaluated the expression is equivalent to the infinitesimal propagation equation provided that one retains the full infinite dimensional space of spatial degrees of freedom. The derivation is done for a single photon and the resulting expression is generalized to cases for two photons where either one photon or both photons propagate through turbulence. This equivalence between the expression in Lindblad form and the IPE implies that the solution of the full IPE represents a valid density matrix. In practice one may need to truncate the IPE to a finite number of basis elements. The resulting density matrix is therefore not in general properly normalized. Such a result would only represent a valid density matrix in case all the excluded elements are identically zero.

As part of the derivation, we show that the single phase screen model can be expressed by an operator product expansion, which implies that it represents a valid quantum process. We discuss the conditions for weak scintillation and point out that there are situations where the quantum state may retain a non zero entanglement well into the strong scintillation region. Under such condition the single phase screen model would not be valid anymore.

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