Vitality of optical vortices

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ABSTRACT

Optical vortices are always created or annihilated in pairs with opposite topological charges. However, the presence of such a vortex dipole does not directly indicate whether they are associated with a creation or an annihilation event. Here we propose a method to distinguish between vortex dipoles that have just been created and those that are about to be annihilated. We use first and second order transverse derivatives of the optical field to construct a quantity that reveals the nature of the dipoles. Numerical examples are provided as demonstration of the method.

Keywords: Optical vortex dipole, vortex pair annihilation, vortex pair creation, vorticity

1. INTRODUCTION

An optical speckle field[1,2] is a random field in the sense that the angular spectrum of such a speckle field is a completely random, normally distributed complex function, weighed by some envelop function. One can make the optical field slightly less random by setting up some correlations in the angular spectrum. The simplest way to do this is to produce multiple laterally shifted copies of the same random angular spectrum and to add them coherently. The resulting optical field is still random in a certain sense, but the correlations produce interesting phenomena, such as transient dynamics in the evolution of the statistical properties of the optical field. Stochastic optical fields can be better understood by studying such transient dynamics.

Some of the quantities that reveal interesting properties of stochastic optical fields are the optical vortex density[3–12] and, related to that, the topological charge density. Optical vortices are points where the complex function of the optical field is zero. The phase at such points is undefined (singular). Around such singular points the wave front has a helical structure. The associated handedness of the helix gives rise to a topological charge of $+1$ or -1. Theoretically one can have topological charges of larger magnitudes, but in stochastic optical fields such higher order zeros are unstable and they quickly decay into those with topological charge ± 1.

The vortex density and the topological charge density also exhibit transient behavior in the propagation of stochastic optical fields.[13–15] To understand this behavior, one can start by formulating some equations for these quantities, based on the topological properties of optical vortices. Topological charge is locally conserved: the net flow of topological charge into a finite region of space is zero. For a propagating optical field this implies that optical vortices must be created or destroyed in neutral combinations — pairs of oppositely charged vortices. Apart from these creation or annihilation events, the vortex distributions are conserved. Therefore, the positive and negative vortices must separately obey the following conservation equations[13]

$$\partial_z n_p(\mathbf{x}) + \nabla\cdot\mathbf{J}_p(\mathbf{x}) = \mathcal{C}(\mathbf{x}) - \mathcal{A}(\mathbf{x}) \quad (1)$$

$$\partial_z n_n(\mathbf{x}) + \nabla\cdot\mathbf{J}_n(\mathbf{x}) = \mathcal{C}(\mathbf{x}) - \mathcal{A}(\mathbf{x}), \quad (2)$$

where $\partial_z = \partial/\partial z$; $n_p(\mathbf{x})$ and $n_n(\mathbf{x})$ are, respectively, the positive and negative vortex densities; $\mathbf{J}_p(\mathbf{x})$ and $\mathbf{J}_n(\mathbf{x})$ are transverse currents for the positive and negative vortices, respectively; ∇ is the two-dimensional transverse nabla operator; and $\mathcal{C}(\mathbf{x})$ and $\mathcal{A}(\mathbf{x})$ are, respectively, the local expectation values for the number of creation and annihilation events per unit volume.

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The vortex density and the topological charge density are, respectively, given by the sum and difference of the distributions of positive and negative vortices. One can therefore also express these conservation equations as

$$\begin{aligned} \partial_z V(\mathbf{x}) + \nabla \cdot \mathbf{J}_V(\mathbf{x}) &= 2[\mathcal{C}(\mathbf{x}) - \mathcal{A}(\mathbf{x})] && (3) \\ \partial_z T(\mathbf{x}) + \nabla \cdot \mathbf{J}_T(\mathbf{x}) &= 0, && (4) \end{aligned}$$

where $V(\mathbf{x}) = n_p(\mathbf{x}) + n_n(\mathbf{x})$ and $T(\mathbf{x}) = n_p(\mathbf{x}) - n_n(\mathbf{x})$ are, respectively, the vortex density and the topological charge density, and $\mathbf{J}_V(\mathbf{x})$ and $\mathbf{J}_T(\mathbf{x})$ are transverse currents associated with the vortex density and the topological charge density, respectively. The exact nature or composition of these currents is not currently known. This is one of the main challenges in uncovering the dynamical behavior of these quantities.

The right-hand side of Eq. (4) is zero, expressing the conservation of topological charge — creation and annihilation events do not affect the local topological charge. Due to its simpler form, Eq. (4) is the obvious choice to tackle first, and indeed some work has been done on mechanisms associated with the divergence of the topological charge current.[13–15]

The equation in Eq. (3) is more challenging due to the non-zero right-hand side, which represents the net gain in the number of vortices due to the creation and annihilation of vortex pairs. Note that the factor 2 on the right-hand side of Eq. (3) implies that creation and annihilation events always involve 2 vortices each. Note also that we need the difference between the density of creation events and the density of annihilation events — the evolution of the vortex density in stochastic optical fields is related to the difference in the rates of pair creation and annihilation. For the sum of the two kinds of events, it would be enough to compute the density of critical points. The latter can either represent creation or annihilation events. For the difference, on the other hand, one must be able to identify whether a critical point is an annihilation event or whether it is a creation event. Hence, the ability to distinguish between pair creation and annihilation events is necessary to understand the evolution of the vortex densities.

Here we derive a quantity that distinguishes between these two types of critical points and we call this quantity the vortex *vitality*, because it is associated with the birth or death of optical vortices.

The vitality of optical vortices may also have some other more direct applications. If one can measure it, one could potentially control it. Hence, it may be possible to construct a feedback control system that can force the vitality in an optical field to be such that all vortices would tend to be annihilated and none would be created. In such a case the number of optical vortices in the optical field would be reduced. This can be useful in free-space applications where propagation through atmospheric turbulence can cause strong scintillation, which produces unwanted optical vortices in the optical beam.

2. THEORY

For any particular point in an optical field to coincide with an annihilation event or a creation event, the point must have the following properties:

In the first place, it must lie on a vortex line. Such vortex line are found where the complex field is zero. It turns out that this prerequisite is unnecessary. As we will see, the final expression only contain derivatives of the optical field. Therefore, even if the point does not lie on a vortex line, one could shift a vortex line to that point by adding an appropriate constant field to the optical field.

In three-dimensional space, optical vortices appear as directed lines, where the direction represents topological charge flow. The topological charge flow is given by the vorticity,[12]

$$\Omega = \frac{\mathrm{i}}{2} \nabla g(\mathbf{x}) \times \nabla g^*(\mathbf{x}), \tag{5}$$

where $g(\mathbf{x})$ is the optical field and $g^*(\mathbf{x})$ is its complex conjugate.

Secondly, it must be a critical point — i.e. the vortex line must change direction with respect to the propagation axis. Let's assume the propagation direction is the z-axis. The sign of the z-component of the vorticity then represents the topological charge of a vortex in the plane perpendicular to the propagation direction. Points where the vortex line changes direction also represent points where two oppositely charged vortices meet. Since the vorticity is a vector that is tangential to the vortex line, the point where the line turns around coincides with the point where the z-component of the vorticity becomes zero. Hence, critical points are indicated by the condition

$$\Omega \cdot \hat{z} = \frac{\mathrm{i}}{2}(g_x g_y^* - g_y g_x^*) = 0, \tag{6}$$

where the subscripts represent derivatives.

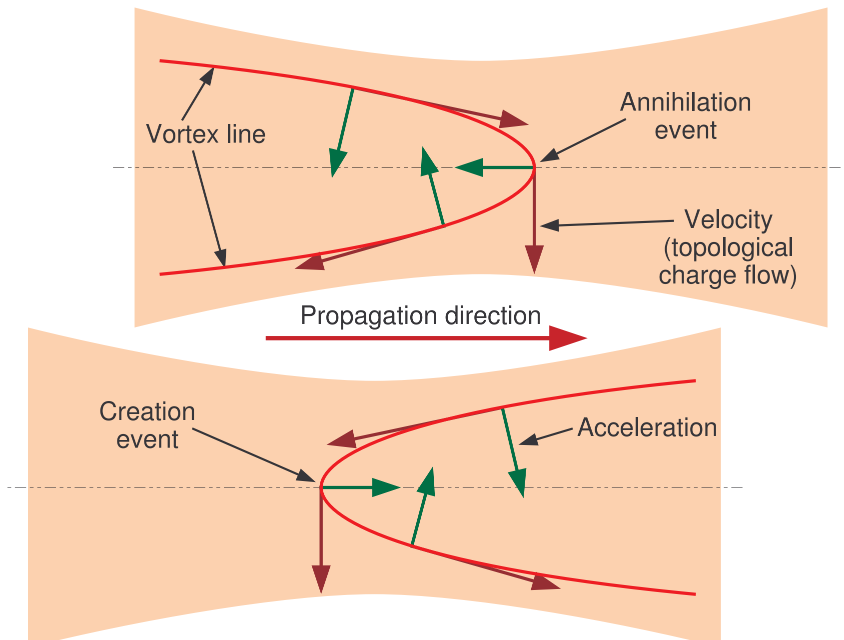

Figure 1. Diagram of optical fields containing vortex lines that either curve back to form an annihilation event or curve forward to produce a creation event. The velocity vectors (topological charge flow vectors) and the acceleration vectors are, respectively, shown as the dark red and dark green arrows.

Thirdly, one wants to distinguish between annihilation events and creation events. For an annihilation event, the vortex line curves backward and for a creation event, the vortex line curves forward. One can quantify this difference by considering the acceleration vector. First we define the 'velocity' vector as the normalized vorticity

$$\mathbf{v} = \frac{\Omega}{|\Omega|}. \tag{7}$$

Assuming that the velocity vector is parameterized by a variable t, so that

$$\mathbf{v}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}, \tag{8}$$

one can compute the acceleration vector as

$$\mathbf{a}(t) = \partial_t \mathbf{v}(t) = \partial_t x(t)\hat{x} + \partial_t y(t)\hat{y} + \partial_t z(t)\hat{z}. \tag{9}$$

where $\partial_t = \partial/\partial t$. The sign of the z-component of the acceleration vector tells us whether the vortex line curves forward or backward (see Fig. 1). It can therefore be used to distinguish between creation and annihilation events. If the z-component of the acceleration vector at a critical point is positive (negative) — pointing in the same direction as (opposite direction to) the propagation vector — then the critical point represents a creation (an annihilation) event. Hence, the z-component of the acceleration vector represents the optical vortex dipole *vitality*.

Expressed in terms of the transverse derivatives of the optical field, the vitality (at a critical point) is[16]

$$\mathcal{V} = H_1 H_3 + H_2 H_4, \tag{10}$$

where

$$H_1 = g_{xx}^* g_y + g_{xx} g_y^* + g_{yy}^* g_y + g_{yy} g_y^* \tag{11}$$
$$H_2 = g_{xx}^* g_x + g_{xx} g_x^* + g_{yy}^* g_x + g_{yy} g_x^* \tag{12}$$
$$H_3 = \mathrm{i}\left(g_{xx}^* g_y - g_{xx} g_y^* - g_{xy}^* g_x + g_{xy} g_x^*\right) \tag{13}$$
$$H_4 = \mathrm{i}\left(g_{yy}^* g_x - g_{yy} g_x^* - g_{xy}^* g_y + g_{xy} g_y^*\right). \tag{14}$$

The sign of $\mathcal{V}$ indicates whether the critical point is an annihilation event or a creation event.

3. NUMERICAL EXAMPLES

Since they contain numerous vortices that are constantly being annihilated and created, speckle fields are ideal for demonstrating the use of the vitality. A speckle field can be readily simulated numerically by producing a random angular spectrum of limited support. Here we use the following expression for the speckle field

$$\psi(\mathbf{x}) = \sum_n \chi_n \exp(-i\mathbf{k}_n \cdot \mathbf{x}), \tag{15}$$

where χ_n represents random complex coefficients and $\mathbf{k}_n$ represents random propagation vectors restricted to lie within a given cone angle around the z-axis. In the numerical simulation one reconstructs this speckle field at a sequence of z values to obtain a sequence of two-dimensional optical fields showing the evolution of the field. From this evolution one can identify critical points where the vortices either annihilate or are being created in pairs. One can now use Eq. (10) to compute the vitality at every point in the optical field. At critical points, the sign of the vitality indicates whether the critical point represents an annihilation event or a creation event.

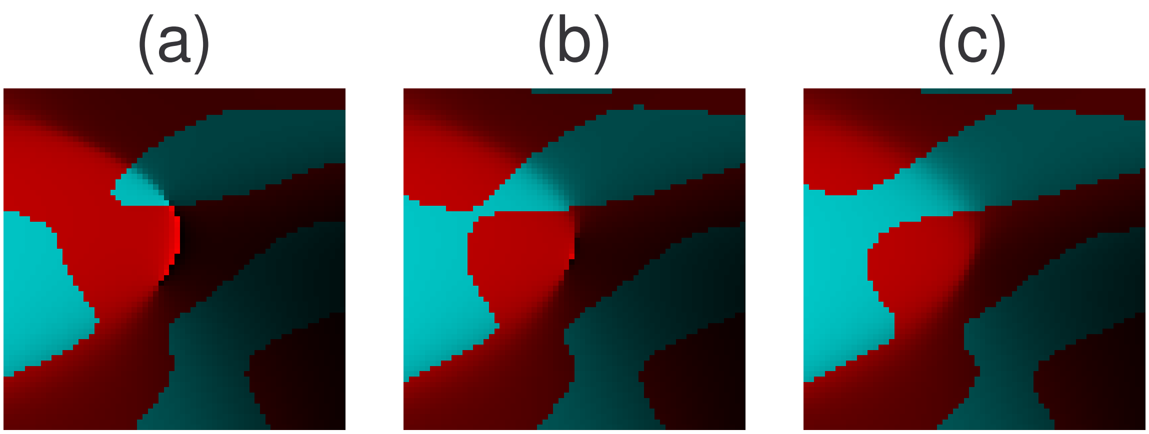

Figure 2. A sequence of three color coded phase functions of an optical speckle field. Regions with positive (negative) vitality are shown in cyan (red). The three images in the sequence denote consecutive slices of the phase of the optical field along the propagation direction, separated by fixed distances along z. The sequence shows how a vortex pair is annihilated in a (red) region of negative vitality.

In Fig. 2 a sequence of three images of the phase of the optical field is shown. These three images show how a pair of optical vortices are annihilated. The sign of the vitality is represented by the color: cyan for positive vitality and red for negative vitality. The sequence of images in Fig. 2 shows that the annihilation occurs in a region of negative vitality.

Another sequence of three images of the phase of the optical field is shown in Fig. 3. In this case the images show the creation of a pair of optical vortices. The images in Fig. 3 show that the creation event occurs in a region of positive vitality.

4. CONCLUSION

To distinguish between vortex dipole creation and annihilation events, we propose a quantity, which we call the vitality, and which is expressed in terms of the transverse first and second order derivatives of the optical field. We use numerically simulated speckle fields to demonstrate that this quantity is successful in identifying the type of critical point.

The expression for the vitality is the first step toward computing the probability density for the difference of creation and annihilation events, appearing in the continuity equation for the vortex density.

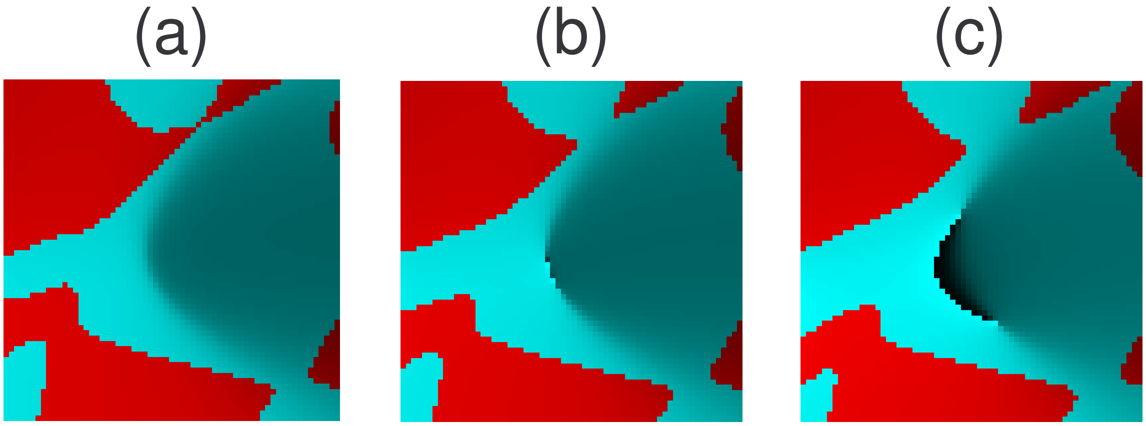

Figure 3. A sequence of three color coded phase functions of an optical speckle field. Regions with positive (negative) vitality are shown in cyan (red). The three images in the sequence denote consecutive slices of the phase of the optical field along the propagation direction, separated by fixed distances along z. The sequence shows how a vortex pair is created in a (cyan) region of positive vitality.

The main purpose of the vitality is to distinguish between vortex dipole creation and annihilation events. Although one can compute the vitality as a three-dimensional continuous function for the whole optical field, the vitality only gives an unambiguous identification of the type of event at the location of a critical point. Critical points are readily identified as points on a vortex line where the z-component of the vorticity is zero and the vortex lines are given by the zeros of the complex optical field.

It is reasonable to argue that the probability of annihilation would increase as the separation distance between oppositely charged vortices decreases while both lie inside a region of negative vitality. If the separation distance is much smaller than the transverse scale (coherence distance) of the optical field, then the probability that the vortex pair would annihilate within a distance on the order of the longitudinal scale (the Rayleigh range), should be fairly high. By taking all these aspects into account, it may be possible to give an estimate of the probability for annihilation, but such an estimate is not known yet.

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