

How to distinguish between the annihilation and the creation of optical vortices

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Optical vortices are always created or annihilated as vortex dipoles — pairs with opposite topological charge. Here a quantity, consisting of the transverse first and second derivatives of the optical field, is derived with which one can distinguish between vortex dipole annihilation and creation events. Analytical and numerical examples are provided as demonstration of the method.

Stochastic optical fields, such as speckle fields, contain numerous dark points where the intensity vanishes and the phase is undefined. These points are phase singularities or optical vortices [1–3]. The statistical properties of optical vortices in speckle fields have been studied extensively [4–13]. The phase around a phase singularity changes by an integer multiple of 2π , resulting in a helical wavefront, which gives rise to a twisted motion in the flow of the optical power. Hence, the term ‘optical vortex.’ The handedness of the phase circulation around the phase singularities allows optical vortices to be separated into either positive or negative vortices. The sign is associated with the topological charge of the optical vortices, which can be any integer. However, only optical vortices with topological charges of $\nu = \pm 1$ are stable in stochastic optical fields. Vortices with higher topological charges quickly decay into those with $\nu = \pm 1$.

Optical vortices are topologically stable and cannot be removed or destroyed by local perturbations in the optical field. They can only be created and annihilated in pairs of opposite topological charge. This property of optical vortices presents a challenge in applications where light propagates through a turbulent atmosphere. If the resulting scintillation is weak, the distortion can be represented by a single random phase modulation, which can be corrected in an adaptive optics system. On the other hand, if the scintillation is strong, the phase distortions will eventually produce optical vortices. In principle one can remove such vortices by multiplying the optical field with the conjugate phase [14, 15]. However, one cannot use the continuous deformable mirror of an adaptive optics system for this. It is necessary to remove the vortices before the remaining continuous phase distortions can be corrected [16]. Part of the problem is to distinguish between vortices that will annihilate by themselves and those that won’t. It would be unreasonable to expect that one can make exact predictions for each and every vortex in a stochastic optical field, but an ability to assign probabilities for the annihilation of vortex pairs would already be helpful.

The evolution of the vortex density in stochastic optical fields is closely related to the difference in the rates

of pair creation and annihilation. As a result, the ability to distinguish between pair creation and annihilation events is vital for an understanding of the evolution of the vortex densities.

In this letter we provide a method to distinguish between annihilation and creation events. The points in an optical field where a pair of optical vortices are created or annihilated are, for the purpose of this paper, referred to as *critical points*. Here we study the properties of the optical fields in the vicinity of critical points and propose a method to distinguish between points of annihilation and those associated with pair creation. Such a distinction may help to identify which pairs of vortices will disappear by themselves and which pairs need additional processing to force them to annihilate. Previously, Freund has identified the topologies of the Poincarè-Hopf indices in the region of critical points [17]. Unfortunately, these topologies do not provide an unambiguous identification of the type of critical point.

To demonstrate the challenge in identifying whether a pair of oppositely charged optical vortices (vortex dipole), is about to be annihilated or has just been created, we start with a simple illustrative example. The analyses of second-order polynomial Gaussian beams [18, 19] allow one to construct examples of beams containing vortex dipoles that will either be annihilated or created in the waist of the beam. The general form of a second-order polynomial Gaussian beam is [20]

$$g(x, y, z) = \frac{P(x, y, z)}{(1 - iz)^3} \exp\left(-\frac{x^2 + y^2}{1 - iz}\right), \quad (1)$$

where the transverse coordinates x and y are measured in units of the beam waist radius and the propagation distance z is measured in units of the Rayleigh range. The prefactor $P(x, y, z)$ is a second-order bivariate polynomial in x and y with complex z -dependent coefficients.

Here we consider two examples where the prefactor either contains a vortex dipole that is annihilated in the beam waist,

$$P_{\text{an}}(x, y, z) = x^2 + y^2 + \frac{\sqrt{3}}{5}(1 + 2i)(1 - iz)y + \frac{11 + 12i}{1060}(1 - iz)(9 + 48i - 53iz) \quad (2)$$

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or contains one that is created in the beam waist,

$$P_{cr}(x, y, z) = x^2 + y^2 - \frac{\sqrt{3}}{5}(4-2i)(1-iz)y - iz(1-iz). \quad (3)$$

In Fig. 1 we compare the phase functions of these two beams, respectively prior to annihilation and just after creation of the vortex dipole. The resulting phase functions both contain a pair for phase singularities. Although the optical fields must contain properties that would cause the one vortex dipole to annihilate and the other one not to annihilate, one can not say which is which from a casual observation of their phase functions. Therefore, a need exists for a method with which this distinction can be made.

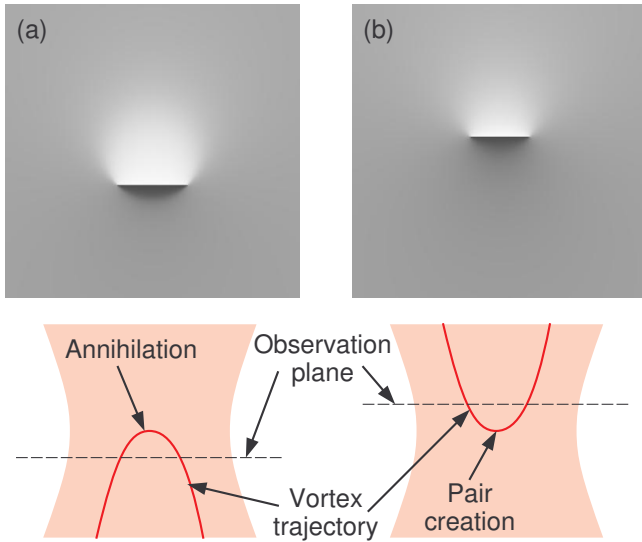


Fig. 1. Comparison of the phase functions of polynomial Gaussian beams that contain vortices dipoles (a) prior to annihilation and (b) after pair creation. The diagrams beneath the phase functions show the respective beams propagating upward, with the locations of the observation planes and the trajectories of the vortex dipoles as indicated.

In the three-dimensional space occupied by a paraxial optical field, the phase singularities appear as directed lines, threading back and forth along the direction of propagation. The direction of the lines are defined by the direction of topological charge flow, which can be defined with a right-hand rule to link the handedness of the phase circulation with the direction of topological charge flow. One can also define the topological charge flow vector using the vorticity [12], given by

$$\begin{aligned} \Omega &= \nabla a(x, y, z) \times \nabla b(x, y, z) \\ &= \frac{i}{2} \nabla g(x, y, z) \times \nabla g^*(x, y, z), \end{aligned} \quad (4)$$

where the complex optical field and its real and imaginary parts are related by $g = a + ib$. For the purpose of the current discussion we define the topological charge

flow vector as the normalized vorticity vector

$$\mathbf{T} = \frac{\Omega}{|\Omega|} = \frac{\nabla a(x, y, z) \times \nabla b(x, y, z)}{|\nabla a(x, y, z) \times \nabla b(x, y, z)|}. \quad (5)$$

Note that the definition of \mathbf{T} only contains derivatives of the optical field and not the optical field itself. As a result, \mathbf{T} is invariant to any constant complex field that may be added to the optical field. The effect of such an added constant field is to shift the vortex trajectories to other locations in the optical field. The new vortex locations are found where the added complex constant cancels the amplitude in the optical field to produce zeros. Every point in the optical field can thus be associated with a vortex trajectory. The topological charge flow is therefore a vector field that is defined at every point in the optical field.

Critical points are located where vortex trajectories turn around. (This is analogous to the sign-change of the C-point polarization index [21].) Provided that the normal vector of the observation planes (for which we use the propagation vector \hat{z}), is defined unambiguously, the locations of these critical points are also unambiguous. At such points the topological charge flow is perpendicular to the propagation direction. Hence, the z -component of the vorticity becomes zero,

$$\Omega_z = a_x b_y - a_y b_x = \frac{i}{2} (g_x g_y^* - g_y g_x^*) = 0, \quad (6)$$

where the subscripts x and y , respectively indicate the x - and y -derivatives of the field. The zeros of Ω_z generally describe planes in the three-dimensional region of the optical field. Critical points are those points where the optical vortex lines cross the planes of zero Ω_z .

These critical points can either represent annihilation or creation events. To distinguish between these types of critical points, we start by noting that the topological charge flow, as defined in Eq. (5), is a tangent vector along the trajectory of the vortex line. As such, for the sake of this discussion, it can be regarded as a velocity vector with a constant magnitude and a varying direction. A change in the direction of such a velocity vector requires an acceleration vector that is perpendicular to the velocity vector. Assuming that the topological charge flow for a particular vortex trajectory is parameterized by a variable t , so that

$$\mathbf{T}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}, \quad (7)$$

one can compute the acceleration vector as

$$\mathbf{a}(t) = \partial_t \mathbf{T}(t) = \partial_t x(t)\hat{x} + \partial_t y(t)\hat{y} + \partial_t z(t)\hat{z}. \quad (8)$$

where $\partial_t = \partial/\partial t$. Since the topological charge flow is a vector field defined for every point in the optical field, the acceleration vector is also a vector field. At any particular point in the optical field the acceleration vector field is associated with the vortex trajectory that passes through that point for the appropriate complex constant that gives a zero at that point.

The sign of the z -component of the acceleration vector (i.e. the component along the direction of the propagation vector) can be used to distinguish between creation and annihilation events. If the z -component of the acceleration vector at a critical point is positive (negative), then it points in the same direction as (opposite direction to) the propagation vector, which means that the critical point represents a creation (an annihilation) event. As the z -component of the acceleration vector gives us information about the births or deaths of vortex dipoles, we'll refer to it as the optical vortex dipole vitality, or just the *vitality*.

To derive an expression for the vitality, we expand the optical field around an arbitrary point on the transverse plane at a fixed value of z as a Taylor series up to second order in the transverse coordinates. Without loss of generality we choose the point at $(x, y) = (0, 0)$ on the plane where $z = 0$. The z -dependence at this point is determined by adding $g_z z$ to the expansion and using the Helmholtz equation to solve for g_z . [A factor of $\exp(-ikz)$ is multiplied with $g(x, y, z)$ before subjecting it to the Helmholtz equation.] The constant term is set equal to zero, because we assume that the point contains a vortex and the acceleration is invariant to such a constant. The resulting expansion is given by

$$g(x, y, z) = g_x x + g_y y + \frac{1}{2} g_{xx} x^2 + g_{xy} xy + \frac{1}{2} g_{yy} y^2 - i \frac{g_{xx} + g_{yy}}{k} z. \quad (9)$$

We now use Eq. (9) to compute the z -component of the acceleration (the vitality). The result is

$$a_z = \frac{2k [(A_1 H_0 - A_2 H_5) \Omega_z + (H_3 H_1 - H_4 H_2) A_0]}{(4k^2 \Omega_z^2 + A_0)^2}, \quad (10)$$

where Ω_z is given in Eq. (6) and

$$H_0 = a_{xx} a_{xy} + a_{yy} a_{xy} + b_{xx} b_{xy} + b_{yy} b_{xy} \quad (11a)$$

$$H_1 = a_y a_{xx} + a_y a_{yy} + b_y b_{yy} + b_y b_{xx} \quad (11b)$$

$$H_2 = b_x b_{xx} + a_x a_{xx} + b_x b_{yy} + a_x a_{yy} \quad (11c)$$

$$H_3 = a_y b_{xx} - a_x b_{xy} + b_x a_{xy} - b_y a_{xx} \quad (11d)$$

$$H_4 = a_y b_{xy} - a_x b_{yy} + b_x a_{yy} - b_y a_{xy} \quad (11e)$$

$$H_5 = a_{xx}^2 + b_{xx}^2 - a_{yy}^2 - b_{yy}^2 \quad (11f)$$

$$A_0 = H_1^2 + H_2^2 \quad (11g)$$

$$A_1 = H_1^2 - H_2^2 \quad (11h)$$

$$A_2 = H_1 H_2, \quad (11i)$$

with the subscripts x and y representing derivatives as in Eq. (6) and with the real and imaginary parts of the derivatives of the optical field g_{\square} given in terms of a_{\square} and b_{\square} , such that $g_{\square} = a_{\square} + ib_{\square}$.

Although the acceleration vector is defined for every point in the optical field, it is only at critical points ($\Omega_z = 0$) where the sign of its z -component distinguishes between creation and annihilation events. When we set

$\Omega_z = 0$ in Eq. (10), it simplifies to

$$a_z = \frac{2k(H_3 H_1 - H_4 H_2)}{(H_1^2 + H_2^2)}. \quad (12)$$

The sign of the vitality is then given by the sign of $(H_3 H_1 - H_4 H_2)$, indicating whether the critical point represents an annihilation event or a creation event:

$$\begin{aligned} \text{annihilation event} &\Rightarrow a_z < 0 \\ \text{creation event} &\Rightarrow a_z > 0. \end{aligned} \quad (13)$$

To test whether the proposed expression for the vitality in Eq. (10) can successfully distinguish between annihilation and creation events, we apply it to the second-order polynomial Gaussian beams, with prefactors given in Eqs. (2) and (3). All the critical points in these two beams (five in total) are correctly distinguished based on the sign of the vitality. We also applied the vitality to numerical simulations of speckle fields and obtained the same successful results.

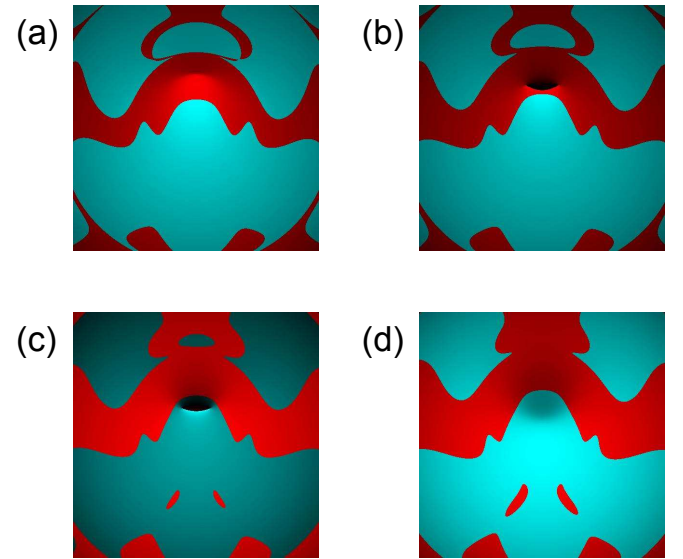


Fig. 2. A sequence of four color coded phase functions of a polynomial Gaussian beam is shown. Regions with positive (negative) vitality are shown in red (turquoise). The four images in the sequence denote consecutive slices of the phase of the beam along the propagation direction, separated by a tenth of a Rayleigh range ($\delta z = z_R/10$). The phase function in (a) represents a case just prior to the creation of a vortex pair. In (b) the vortex pair has just been created in the positive (red) region. The vortices crossed the boundary in (c), moving in a negative (turquoise) region. In (d) we see the phase function just after the vortices annihilated in the negative (turquoise) region.

A sequence of color-coded phase functions, taken from the beam associated with Eq. (2), is shown in Fig. 2, demonstrating that the vitality is successful in distinguishing between creation and annihilation events. Creations always appear in positive regions and annihilations always appear in negative regions.

Due to the continuity of the vitality in Eq. (10) as a function in three dimensions, we extended the identification of the event at a critical point in Fig. 2 to a region surrounding that critical point where the vitality has the same sign as at the critical point. In this way we used such an extension to associate the type of event with the vortex dipole that exists before or after the event while they lie in a region with the same sign. This may suggest that one can use the sign of a region in which a vortex dipole is located to predict whether this dipole will annihilate or not. However, such a prediction can at best be made with a certain probability of success. The sizes of regions on the transverse plane with a particular sign are often relatively small compared to the typical separation distances between vortices. On the other hand, these regions tend to extend relatively far along the propagation direction. The regions change slower along the z -direction than the movement of the vortices, so that vortices tend to move across boundaries between regions. Therefore, when two vortices with opposite charges appear in the same region (with a negative sign) one cannot conclude that their annihilation is imminent. Occasionally they may linger for a while and even move away into other regions. Moreover, in practical applications, noise would add some uncertainty to the value of the vitality, which one would need to take into consideration before making any predictions.

It is reasonable to argue that the probability of annihilation would increase as the separation distance between oppositely charged vortices decreases while both lie inside a region of negative vitality. If the separation distance is much smaller than the transverse scale (coherence distance) of the optical field, then the probability that the vortex pair would annihilate within an distance on the order of the longitudinal scale (the Rayleigh range), should be fairly high. By taking all these aspects into account, it may be possible to give an estimate of the probability for annihilation, but such an estimate is not known yet.

The main purpose of the vitality is to distinguish between vortex dipole creation and annihilation events. Although one can compute the vitality as a three-dimensional continuous function for the whole optical field, the vitality only gives an unambiguous identification of the type of event at the location of a critical

point. Critical points are readily identified as points on a vortex line where $\Omega_z = 0$ and the vortex lines are given by the zeros of the complex optical field.

In conclusion, we derived an expression in terms of the transverse first and second derivatives of the optical field, with which one can distinguish between vortex dipole creation and annihilation events. The successful application of this quantity, which we call the vitality, is demonstrated for the case of second-order polynomial Gaussian beams.

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