Module-theoretic properties of reachability modules for $\mathcal{SRIQ}$

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Abstract. In this paper we investigate the module-theoretic properties of $\bot-$ and $\top$-reachability modules in terms of inseparability relations for the DL $\mathcal{SRIQ}$. We show that, although these modules are not depleting or self-contained, they share the robustness properties of syntactic locality modules and preserve all justifications for an entailment.

1 Introduction

Modularization plays an important part in the design and maintenance of large scale ontologies. Modules are loosely defined as subsets of ontologies that cover some topic of interest, where the topic of interest is defined by a set of symbols. Extracting minimal modules is computationally expensive and even undecidable for expressive DLs [3, 4]. Therefore, the use of approximation techniques and heuristics plays an important role in the efficient design of algorithms. Syntactic locality [3, 4], because of its excellent model theoretic properties, has become an ideal heuristic and is widely used in a diverse set of algorithms [14, 2, 5].

Suntisrivaraporn [14] showed that, for the DL $\mathcal{EL}^+$, $\bot$-locality module extraction is equivalent to the reachability problem in directed hypergraphs. Nortjé et al. [10, 11] extended the reachability problem to include $\top$-locality and introduced bidirectional reachability modules as a subset of $\bot\top^*$-locality modules. This work was further extended to the DL $\mathcal{SROIQ}$ Nortje et al. [12] who showed that extracting $\bot\top^*$-reachability modules is equivalent to extracting frontier graphs in hypergraphs. Reachability modules are not only of importance in hypergraph-based reasoning support for TBoxes [12], but are potentially smaller than syntactic locality modules.

In this paper we investigate the module-theoretic properties of reachability modules for the DL $\mathcal{SRIQ}$. We show that these modules are not self-contained or depleting but they are robust under vocabulary restrictions, vocabulary extensions, replacement and joins. By showing that reachability modules preserve all justifications for entailments, we show that depleting modules are sufficient for preserving all justifications but not necessary.

In Section 2 we give a brief introduction to the DL $\mathcal{SRIQ}$ and modularization as defined by inseparability relations. Section 3 introduces a normal form for $\mathcal{SRIQ}$ TBoxes as well as the rules necessary to transform any such TBox to normal form. In Section 4 we introduce both $\bot$- and $\top$ reachability modules and investigate all their module theoretic properties in terms of inseperability relations. All proofs of the work presented appears in the accompanying appendix. Lastly in Section 5 we conclude this paper with a short summary of the results.

2 Background

In this section we give a brief introduction to modularization and the DL $\mathcal{SRIQ}$ [7] with its syntax and semantics listed in Table 2. N_C and N_R denote disjoint sets of atomic concept names and role names. The set N_R includes the universal role whilst N_C excludes the $\top$ and $\bot$ concepts. For a complete definition of $\mathcal{SRIQ}$, refer to Horrocks et al. [7], and for Description Logics refer to [1].

Constructs	Syntax	Semantics
atomic concept	C	$C^{\mathcal{I}} \in \Delta^{\mathcal{I}}, C \in N_C$
role	R	$R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}, R \in N_R$
inverse role	R^-	$R^{-\mathcal{I}} = \{(y,x) \mid (x,y) \in R^{\mathcal{I}}\}, R \in N_R$
universal role	U	$U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
role composition	$R_1 \circ \ldots \circ R_n$	$\{(x,z) \mid (x,y_1) \in R_1^{\mathcal{I}} \wedge (y_1,y_2) \in R_2^{\mathcal{I}} \wedge \ldots$ $\wedge (y_n,z) \in R_n^{\mathcal{I}}, n \geq 2, R_i \in N_R\}$
top	$\top$	$\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$
bottom	$\bot$	$\bot^{\mathcal{I}} = \emptyset$
negation	$\neg C$	$(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
conjunction	$C_1 \sqcap C_2$	$(C_1 \sqcap C_2)^{\mathcal{I}} = C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
disjunction	$C_1 \sqcup C_2$	$(C_1 \sqcup C_2)^{\mathcal{I}} = C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$
exist restriction	$\exists R.C$	$\{x \mid (\exists y)[(x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}]\}$
value restriction	$\forall R.C$	$\{x \mid (\forall y)[(x,y) \in R^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}]\}$
self restriction	$\exists R.Self$	$\{x \mid (x,x) \in R^{\mathcal{I}}\}$
atmost restriction	$\leqslant nR.C$	$\{x \mid \#\{y \mid (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \leqslant n\}$
atleast restriction	$\geqslant nR.C$	$\{x \mid \#\{y \mid (x,y) \in R^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \geqslant n\}$
Axiom	**Syntax**	**Semantics**
concept inclusion	$C_1 \sqsubseteq C_2$	$C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$
role inclusion	$R_1 \circ \ldots \circ R_n \sqsubseteq R_{n+1}$	$(R_1 \circ \ldots \circ R_n)^{\mathcal{I}} \subseteq R^{\mathcal{I}}, n \geq 1$
reflexivity	$Ref(R)$	$\{(x,x) \mid x \in \Delta^{\mathcal{I}}\} \subseteq R^{\mathcal{I}}$
irreflexivity	$Irr(R)$	$\{(x,x) \mid x \in \Delta^{\mathcal{I}}\} \cap R^{\mathcal{I}} = \emptyset$
disjointness	$Dis(R,S)$	$S^{\mathcal{I}} \cap R^{\mathcal{I}} = \emptyset$

Table 1. Syntax and semantics of $\mathcal{SRIQ}$

Module extraction is the process of extracting subsets of axioms from TBoxes that are self contained with respect to some criteria. These sets of axioms, called *modules*, may be used for various purposes such as reuse, optimization and error pinpointing amongst others [4, 14].

Definition 1. (Module for the arbitrary DL $\mathcal{L}$) *Let $\mathcal{L}$ be an arbitrary description language, $\mathcal{O}$ an $\mathcal{L}$ ontology, and σ a statement formulated in $\mathcal{L}$. Then,*

$\mathcal{O}' \subseteq \mathcal{O}$ *is a* module for σ in $\mathcal{O}$*(a* σ-module in $\mathcal{O}$*) whenever:* $\mathcal{O} \models \sigma$ *if and only if* $\mathcal{O}' \models \sigma$.

Definition 1 is sufficiently general so that any subset of an ontology preserving a statement of interest is considered a module, the entire ontology is therefore a module in itself.

Different use cases usually result in different notions of what the definition and characteristics of a module should be. Modules are often defined via the notion of conservative extensions. Given some signature (a set of concept and role names) and a set of axioms, a conservative extension of this set is simply one that implies all the same consequences over the signature. More formally:

Definition 2. (Conservative extension [4]) *Let* $\mathcal{T}$ *and* $\mathcal{T}_1$ *be two TBoxes such that* $\mathcal{T}_1 \subseteq \mathcal{T}$*, and let* Σ *be a signature. Then*

- $\mathcal{T}$ *is a* Σ*-conservative extension of* $\mathcal{T}_1$ *if, for every* α *with* $Sig(\alpha) \subseteq \Sigma$*, we have* $\mathcal{T} \models \alpha$ *iff* $\mathcal{T}_1 \models \alpha$.
- $\mathcal{T}$ *is a conservative extension of* $\mathcal{T}_1$ *if* $\mathcal{T}$ *is a* Σ*-conservative extension of* $\mathcal{T}_1$ *for* $\Sigma = Sig(\mathcal{T}_1)$.

Given that both sets of axioms imply the same consequences for a given signature we may then use the smaller set whenever we wish to reason over this signature. A closely related notion to conservative extensions is that of *inseparability.*

Definition 3. *[13]* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are* Σ*-concept name inseparable, written* $\mathcal{T}_1 \equiv^c_\Sigma \mathcal{T}_2$*, if for all* Σ*- concept names* C, D*, it holds that* $\mathcal{T}_1 \models C \sqsubseteq D$ *if and only if* $\mathcal{T}_2 \models C \sqsubseteq D$.

Definition 4. *[13]* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are* Σ*-subsumption inseparable, written* $\mathcal{T}_1 \equiv^s_\Sigma \mathcal{T}_2$*, if for all terms* X, Y *that are concepts or roles over* Σ*, it holds that* $\mathcal{T}_1 \models X \sqsubseteq Y$ *if and only if* $\mathcal{T}_2 \models X \sqsubseteq Y$.

Definition 5. *[13] Let* $\mathcal{T}$ *be a TBox,* $\mathcal{M} \subseteq \mathcal{T}$*,* S *an inseparability relation and* Σ *a signature. We call* $\mathcal{M}$

- *an* S_Σ*-module of* T *if* $\mathcal{M} \equiv^S_\Sigma \mathcal{T}$.
- *a self-contained* S_Σ*-module of* $\mathcal{T}$ *if* $\mathcal{M} \equiv^S_{\Sigma \cup Sig(\mathcal{M})} \mathcal{T}$.
- *a depleting* S_Σ*-module of* $\mathcal{T}$ *if* $\emptyset \equiv^S_{\Sigma \cup Sig(\mathcal{M})} \mathcal{T} \setminus \mathcal{M}$.

Modules may therefore be characterized by some inseparability criteria. It is of interest how modules defined this way would behave under different use case scenarios. For this purpose, several properties of inseparability relations [8] have been investigated in the literature, which allows us to compare different definitions of modules. Given a TBox $\mathcal{T}$ and a module $\mathcal{M} \subseteq \mathcal{T}$ for a signature Σ, we are interested in the following inseparability properties:

- *Robustness under vocabulary restrictions* implies that when we wish to restrict the symbols from Σ further we do not need to import a different module and may continue to use $\mathcal{M}$.
- *Robustness under vocabulary extension* implies that should we wish to add new symbols to Σ that do not appear in $\mathcal{T}$ we do not need to use a different module but may use $\mathcal{M}$.
- *Robustness under replacement* ensures that the result of importing $\mathcal{M}$ into a TBox $\mathcal{T}_1$ is a module of the result of importing $\mathcal{T}$ into $\mathcal{T}_1$. This is also called module coverage and refers to the fact that importing a module does not affect its property of being a module.
- *Robusness under joins* implies that if $\mathcal{T}$ and $\mathcal{T}_1$ are inseparable w.r.t. Σ and all the terms they share are from Σ, then each of them are inseparable with their union w.r.t. Σ.

More formally:

Definition 6. *[8] The inseparability relation S is called*

- *robust under vocabulary restrictions if, for all TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$ and all signatures Σ, Σ' with $\Sigma \subseteq \Sigma'$, the following holds: if $\mathcal{T}_1 \equiv^S_{\Sigma'} \mathcal{T}_2$, then $\mathcal{T}_1 \equiv^S_{\Sigma} \mathcal{T}_2$.*
- *robust under vocabulary extensions if, for all TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$ and all signatures Σ, Σ' with $\Sigma' \cap (Sig(\mathcal{T}_1) \cup Sig(\mathcal{T}_2)) \subseteq \Sigma$, the following holds: if $\mathcal{T}_1 \equiv^S_{\Sigma} \mathcal{T}_2$, then $\mathcal{T}_1 \equiv^S_{\Sigma'} \mathcal{T}_2$.*
- *robust under replacement if, for all TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$ and all signatures Σ and every TBox $\mathcal{T}$ with $Sig(\mathcal{T}) \cap (Sig(\mathcal{T}_1) \cup Sig(\mathcal{T}_2)) \subseteq \Sigma$, the following holds: if $\mathcal{T}_1 \equiv^S_{\Sigma} \mathcal{T}_2$ then $\mathcal{T}_1 \cup \mathcal{T} \equiv^S_{\Sigma} \mathcal{T}_2 \cup \mathcal{T}$.*
- *robust under joins if, for all TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$ and all signatures Σ with $Sig(\mathcal{T}) \cap Sig(\mathcal{T}_2) \subseteq \Sigma$, if $\mathcal{T}_1 \equiv^S_{\Sigma} \mathcal{T}_2$ then $\mathcal{T}_i \equiv^S_{\Sigma} \mathcal{T}_1 \cup \mathcal{T}_2$, for $i = 1, 2$.*

3 Normal Form

In this section we introduce a normal form for $\mathcal{SRIQ}$ TBoxes. We utilize normalization in order to simplify the definitions, to ease the understanding of the work that follows, as well as to simplify the presentation of proofs.

Definition 7. *Given $B_i \in (N_C \cup \{\top\})$, $C_i \in (N_C \cup \{\bot\})$, $D \in \{\exists R.B, \geq nR.B, \exists R.Self\}$, with R, S, R_i, S_i role names from N_R or their inverses and $n \geqslant 1$, a $\mathcal{SRIQ}$ TBox $\mathcal{T}$ is in* **normal form** *if every axiom $\alpha \in \mathcal{T}$ is in one of the following forms:*

α_1: $B_1 \sqcap \ldots \sqcap B_n \sqsubseteq C_1 \sqcup \ldots \sqcup C_m$	α_2: $D \sqsubseteq C_1 \sqcup \ldots \sqcup C_m$
α_3: $B_1 \sqcap \ldots \sqcap B_n \sqsubseteq D$	α_4: $R_1 \circ \ldots \circ R_n \sqsubseteq R_{n+1}$
α_5: $R_1 \sqsubseteq R_2$	α_6: $D_1 \sqsubseteq D_2$
α_7: $Dis(R_1, R_2)$	

In order to normalize a $\mathcal{SRIQ}$ TBox $\mathcal{T}$ we repeatedly apply the normalization rules from Table 2. Each application of a rule rewrites an axiom into its equivalent normal form. It is easy to see that the application of every rule ensures that the normalized TBox is a conservative extension of the original. We note that the $\mathcal{SRIQ}$ axiom $Ref(R)$ is represented by its equivalent $\top \sqsubseteq \exists R.Self$ and $Irr(R)$ by $\exists R.Self \sqsubseteq \bot$ [1].

Table 2. $\mathcal{SRIQ}$ normalization rules

NR1	$\hat{B} \sqcap \neg\hat{C}_2 \sqsubseteq \hat{C}_1 \rightsquigarrow \hat{B} \sqsubseteq \hat{C}_1 \sqcup \hat{C}_2$
NR2	$\hat{B}_1 \sqsubseteq \hat{C} \sqcup \neg\hat{B}_2 \rightsquigarrow \hat{B}_1 \sqcap \hat{B}_2 \sqsubseteq \hat{C}$
NR3	$\hat{B} \sqcap \hat{D} \sqsubseteq \hat{C} \rightsquigarrow \hat{B} \sqcap A \sqsubseteq \hat{C},\ \hat{D} \sqsubseteq A,\ A \sqsubseteq \hat{D}$
NR4	$\hat{B} \sqsubseteq \hat{C} \sqcup \hat{D} \rightsquigarrow \hat{B} \sqsubseteq \hat{C} \sqcup A,\ \hat{D} \sqsubseteq A,\ A \sqsubseteq \hat{D}$
NR5	$\hat{B} \sqsubseteq \hat{C}_1 \sqcap \hat{C}_2 \rightsquigarrow \hat{B} \sqsubseteq \hat{C}_1,\ \hat{B} \sqsubseteq \hat{C}_2$
NR6	$\hat{B}_1 \sqcup \hat{B}_2 \sqsubseteq \hat{C} \rightsquigarrow \hat{B}_1 \sqsubseteq \hat{C},\ \hat{B}_2 \sqsubseteq \hat{C}$
NR7	$\ldots \forall R.\hat{C} \ldots \rightsquigarrow \ldots \neg\exists R.A \ldots,\ A \sqcap \hat{C} \sqsubseteq \bot,\ \top \sqsubseteq A \sqcup \hat{C}$
NR8	$\ldots \exists R.\hat{D} \ldots \rightsquigarrow \ldots \exists R.A \ldots,\ \hat{D} \sqsubseteq A,\ A \sqsubseteq \hat{D}$
NR9	$\ldots \geqslant nR.\hat{D} \ldots \rightsquigarrow \ldots \geqslant nR.A \ldots,\ \hat{D} \sqsubseteq A,\ A \sqsubseteq \hat{D}$
NR10	$\ldots \leqslant nR.\hat{C} \ldots \rightsquigarrow \ldots \neg(\geqslant (n+1)R.\hat{C}) \ldots$
NR11	$\hat{B} \equiv \hat{C} \rightsquigarrow \hat{B} \sqsubseteq \hat{C}, \hat{C} \sqsubseteq \hat{B}$
NR12	$\geqslant 0R.B \sqsubseteq \hat{C} \rightsquigarrow \top \sqsubseteq \hat{C}$
NR13	$\hat{B} \sqsubseteq \exists R.\bot \rightsquigarrow \hat{B} \sqsubseteq \bot$
NR14	$\hat{B} \sqsubseteq \geqslant nR.\bot \rightsquigarrow \hat{B} \sqsubseteq \bot$
NR15	$\hat{B} \sqsubseteq \geqslant 0R.B \rightsquigarrow$
NR16	$\geqslant nR.\bot \sqsubseteq \hat{C} \rightsquigarrow$
NR17	$\exists R.\bot \sqsubseteq \hat{C} \rightsquigarrow$
NR18	$\hat{B} \sqcap \bot \sqsubseteq \hat{C} \rightsquigarrow$
NR19	$\bot \sqsubseteq \hat{C} \rightsquigarrow$
NR20	$\hat{B} \sqsubseteq \hat{C} \sqcup \top \rightsquigarrow$
NR21	$\hat{B} \sqsubseteq \top \rightsquigarrow$

Above A is a new concept name not in N_C, $\hat{B}_i$ and $\hat{C}_i$ are possibly complex concept descriptions and $\hat{D}$ a complex concept description. $R \in N_R$ or it's inverse, $n \geqslant 0$

Theorem 1. *Exhaustively applying the rules from Table 2 to any $\mathcal{SRIQ}$ TBox $\mathcal{T}$ results in a $\mathcal{SRIQ}$ TBox $\mathcal{T}'$ in normal form. The normalization process can be completed in linear time in the number of axioms.*

Example 1. Let $\alpha_1 = B \sqsubseteq \neg C$, and $\alpha_2 = \neg A \sqsubseteq B$. Then, α_1 may be normalized by application of rule NR2 to $\alpha_1^N = B \sqcap C \sqsubseteq \bot$ since $\neg C = \neg C \sqcup \bot$. α_2 may be normalized by application of rule NR1 to $\alpha_2^N = \top \sqsubseteq B \cup A$ since $\neg A = \neg A \sqcap \top$.

For the rest of this paper we assume that all TBoxes are in normal form.

4 Reachability Modules

Deciding conservative extensions has been shown to be computationally expensive or even undecidable for relatively inexpressive DLs. Therefore, an approximation of these modules, based on syntax, called syntactic locality modules [4] has been introduced. Given a normalized TBox $\mathcal{T}$, the definition of syntactic locality can be simplified to the following:

Definition 8. (Normalized Syntactic Locality) *Let Σ be a signature and $\mathcal{T}$ a normalized $\mathcal{SRIQ}$ TBox. An axiom α is $\bot$-local w.r.t. Σ ($\top$-local w.r.t Σ) if $\alpha \in Ax(\Sigma)^{\bot}$ ($\alpha \in Ax(\Sigma)^{\top}$), as defined in the grammar:*

$\bot$-Locality	
$\mathbf{Ax}(\mathbf{\Sigma})^{\bot}$	$::= C^{\bot} \sqsubseteq C \mid w^{\bot} \sqsubseteq R \mid Dis(S^{\bot}, S) \mid Dis(S, S^{\bot})$
$\mathbf{Con}^{\bot}(\mathbf{\Sigma})$	$::= A^{\bot} \mid C^{\bot} \sqcap C \mid C \sqcap C^{\bot} \mid \exists R^{\bot}.C \mid \exists R.C^{\bot} \mid \exists R^{\bot}.Self \mid$ $\geqslant nR^{\bot}.C \mid \geqslant nR.C^{\bot}$
$\top$-Locality	
$\mathbf{Ax}(\mathbf{\Sigma})^{\top}$	$::= C \sqsubseteq C^{\top} \mid w \sqsubseteq R^{\top}$
$\mathbf{Con}^{\top}(\mathbf{\Sigma})$	$::= A^{\top} \mid C^{\top} \sqcup C \mid C \sqcup C^{\top} \mid \exists R^{\top}.C^{\top} \mid \geqslant nR^{\top}.C^{\top} \mid$ $\exists R^{\top}.Self$

In the grammar, we have that $A^{\bot}, A^{\top} \notin \Sigma$ is an atomic concept, $R^{\bot}$ (resp. $S^{\bot}$) is either an atomic role (resp. a simple atomic role) not in Σ or the inverse of an atomic role (resp. of a simple atomic role) not in Σ, C is any concept, R is any role, S is any simple role, and $C^{\bot} \in \mathbf{Con}^{\bot}(\mathbf{\Sigma})$, $C^{\top} \in \mathbf{Con}^{\top}(\mathbf{\Sigma})$. We also denote by $w^{\bot}$ a role chain $w = R_1 \circ \ldots \circ R_n$ such that for some i with $1 \leq i \leq n$, we have that R_i is (possibly inverse of) an atomic role not in Σ. A TBox $\mathcal{T}$ is $\bot$-local ($\top$-local) w.r.t. Σ if α is $\bot$-local ($\top$-local) w.r.t. Σ for all $\alpha \in \mathcal{T}$.

Algorithm 1 may be used to extract both the minimal $\bot$ and $\top$-locality based modules for a signature S.

A variant of $\bot$-syntactic locality modules called $\bot$-reachability based modules [14] is based on the reachability problem in directed hypergraphs. Hypergraphs [9, 15] are a generalization of graphs and have been studied extensively since the 1970s as a powerful tool for modelling many problems in Discrete Mathematics. We extend the work done by Nortje et al.[11] and define reachability for $\mathcal{SRIQ}$ TBoxes. We then continue to show that these modules share all the robustness properties of locality modules and therefore is well suited to be used in the ontology reuse scenario.

Definition 9. ($\bot$-Reachability) *Let $\mathcal{T}$ be a $\mathcal{SRIQ}$ TBox in normal form and $\Sigma \subseteq Sig(\mathcal{T})$ a signature. The set of $\bot$-reachable names in $\mathcal{T}$ w.r.t. Σ, denoted by $\Sigma_{\mathcal{T}}^{\leftarrow\bot}$, is defined inductively as follows:*

- *For every $x \in (\Sigma \cup \{\top\})$ we have $x \in \Sigma_{\mathcal{T}}^{\leftarrow\bot}$.*
- *For every inclusion axiom $(\alpha_L \sqsubseteq \alpha_R) \in \mathcal{T}$, if $Sig(\alpha_L) \subseteq \Sigma_{\mathcal{T}}^{\leftarrow\bot}$ then every $y \in Sig(\alpha_R)$ is also in $\Sigma_{\mathcal{T}}^{\leftarrow\top}$.*

Algorithm 1 (Module Extraction Algorithm for $\mathcal{SRIQ}$ [2])

`Procedure extract module`$(\mathcal{O}, S)$

Input: $\mathcal{O}$: ontology; S: signature;
Output: $\mathcal{O}_1$: a module for S in $\mathcal{O}$

$\mathcal{O}_1 := \emptyset, \mathcal{O}_2 := \mathcal{O}$
while not empty$(\mathcal{O}_2)$ **do**
  $\alpha :=$ select axiom$(\mathcal{O}_2)$
  if local$(\alpha,\ S \cup Sig(\mathcal{O}_1))$ **then**
    $\mathcal{O}_2 := \mathcal{O}_2 \setminus \{\alpha\}$
  else
    $\mathcal{O}_1 := \mathcal{O}_1 \cup \{\alpha\}$
    $\mathcal{O}_2 := \mathcal{O} \setminus \mathcal{O}_1$
  end if
end while
return $\mathcal{O}_1$

Every axiom $\alpha := \alpha_L \sqsubseteq \alpha_R$ such that $Sig(\alpha_L) \subseteq \Sigma_{\mathcal{T}}^{\leftarrow\bot}$ we call $\Sigma_{\mathcal{T}}^{\leftarrow\bot}$-reachable. Axioms of the form $Dis(R,S) \in \mathcal{T}$ are $\Sigma_{\mathcal{T}}^{\leftarrow\bot}$-reachable whenever $\{R,S\} \subseteq \Sigma_{\mathcal{T}}^{\leftarrow\bot}$. The set of all $\Sigma_{\mathcal{T}}^{\leftarrow\bot}$-reachable axioms is denoted by $\mathcal{T}_{\Sigma}^{\leftarrow\bot}$ and is called the $\bot$-reachability module for $\mathcal{T}$ over Σ.

It is self-evident from Definition 8 that an axiom is $\bot$-reachable w.r.t Σ exactly when it is not $\bot$-local w.r.t. Σ. Similarly we define an axiom to be $\top$-reachable exactly when it is not $\top$-local.

Definition 10. ($\top$-Reachability) *Let $\mathcal{T}$ be a $\mathcal{SRIQ}$ TBox in normal form and $\Sigma \subseteq Sig(\mathcal{T})$ a signature. The set of $\top$-reachable names in $\mathcal{T}$ w.r.t. Σ, denoted by $\Sigma_{\mathcal{T}}^{\leftarrow\top}$, is defined inductively as follows:*

- *For every $x \in (\Sigma \cup \bot)$ we have that $x \in \Sigma_{\mathcal{T}}^{\leftarrow\top}$.*
- *For all inclusion axioms $(\alpha_L \sqsubseteq \alpha_R) \in \mathcal{T}$, if*
  - *$\alpha_R = \bot$, or*
  - *α_R is of the form $A_1 \sqcup \ldots \sqcup A_n$ and all $A_i \in \Sigma_{\mathcal{T}}^{\leftarrow\top}$, or*
  - *α_R has any other form and there exists some $x \in Sig(\alpha_R) \cap \Sigma_{\mathcal{T}}^{\leftarrow\top}$*

  then every $y \in Sig(\alpha_L)$ is also in $\Sigma_{\mathcal{T}}^{\leftarrow\top}$.

Every axiom $\alpha := \alpha_L \sqsubseteq \alpha_R$ such that, $\alpha_R = \bot$, or α_R is of the form $A_1 \sqcup \ldots \sqcup A_n$ and all $A_i \in \Sigma_{\mathcal{T}}^{\leftarrow\top}$, or α_R has any other form and there exists some $x \in Sig(\alpha_R) \cap \Sigma_{\mathcal{T}}^{\leftarrow\top}$, we call $\Sigma_{\mathcal{T}}^{\leftarrow\top}$-reachable. All axioms of the form $Dis(R,S) \in \mathcal{T}$ are always $\Sigma_{\mathcal{T}}^{\leftarrow\top}$-reachable and $\{R,S\} \subseteq \Sigma_{\mathcal{T}}^{\leftarrow\top}$. The set of all $\Sigma_{\mathcal{T}}^{\leftarrow\top}$-reachable axioms is denoted by $\mathcal{T}_{\Sigma}^{\leftarrow\top}$ and is called the $\top$-reachability module for $\mathcal{T}$ over Σ.

It is easy to show that $\bot$-reachability modules are equivalent to $\bot$-locality modules. However, by the definition of $\top$-reachability we observe that these are not equivalent to $\top$-locality modules.

Example 2. Let $\mathcal{T}$ be a TBox such that $\mathcal{T} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, with $\alpha_1 := A \sqsubseteq \exists r.D_1, \alpha_2 := B \sqsubseteq \geq nr.D_2, \alpha_3 := \exists r.\top \sqsubseteq C, \alpha_4 := D_1 \sqsubseteq D_2$ and let $\Sigma = \{C\}$. Then $\mathcal{T}_\Sigma^{\leftarrow\top} = \{\alpha_1, \alpha_2, \alpha_3\}$ but the $\top$-locality module for $\mathcal{T}$ w.r.t. Σ is $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

The difference stems from the fact that in α_1 and α_2 the $\top$-reachability of r does not ensure the $\top$-reachability of D_1 and D_2 respectively. This occurs because, given an axiom $\alpha = \alpha_L \sqsubseteq \alpha_R$, $\top$-locality ensure that if α is $\top$-local then so are all of the symbols in $Sig(\alpha)$, whereas $\top$-reachability is defined such that the $\top$-reachability of α only guarantees that all symbols of α_L and only some symbols of α_R will be $\top$-reachable. Thus $\top$-reachability based modules are at most the size of $\top$-locality modules but in general could be substantially smaller. Similar to $\bot\top^*$-locality modules we note that reachability module extraction may also be alternated until a fixpoint is reached. These modules are denoted by $\mathcal{T}_\Sigma^{\leftarrow\bot\top^*}$.

In order to investigate the module-theoretic properties of reachability modules, we follow a similar approach to Sattler et al. [13] and define inseparability different from that of conservative extensions. We say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are inseparable if their modules are equivalent, that is, a module extraction algorithm returns the same output for each of them. We define the following inseparability relations for reachability modules:

Definition 11. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be TBoxes and Σ a signature. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are:*

- *$\Sigma - \top$ reachability inseparable, denoted by $\mathcal{T}_1 \equiv_\Sigma^\top \mathcal{T}_2$, if ${\mathcal{T}_1}_\Sigma^{\leftarrow\top} = {\mathcal{T}_2}_\Sigma^{\leftarrow\top}$;*
- *$\Sigma - \bot$ reachability inseparable, denoted by $\mathcal{T}_1 \equiv_\Sigma^\bot \mathcal{T}_2$, if ${\mathcal{T}_1}_\Sigma^{\leftarrow\bot} = {\mathcal{T}_2}_\Sigma^{\leftarrow\bot}$;*
- *$\Sigma - \bot\top^*$ reachability inseparable, denoted by $\mathcal{T}_1 \equiv_\Sigma^{\bot\top^*} \mathcal{T}_2$, if ${\mathcal{T}_1}_\Sigma^{\leftarrow\bot\top^*} = {\mathcal{T}_2}_\Sigma^{\leftarrow\bot\top^*}$.*

Firstly we show that $\top$-reachability modules are subsumption inseparable. Concept inseparability follows as a special case of subsumption inseparability.

Lemma 1. *Let $\mathcal{T}$ be a $\mathcal{SRIQ}$ TBox, and $\Sigma \subseteq Sig(\mathcal{T})$ a signature. Then $\mathcal{T} \models C \sqsubseteq D$ if and only if $\mathcal{T}_\Sigma^{\leftarrow\top} \models C \sqsubseteq D$ for arbitrary $\mathcal{SRIQ}$ concept descriptions C and D such that $Sig(C) \cup Sig(D) \subseteq \Sigma$.*

Corollary 1. *Let $\mathcal{T}$ be a normalized $\mathcal{SRIQ}$ TBox, $\Sigma \subseteq Sig(\mathcal{T})$ a signature and S an inseparability relation from Definitions 3 and 4. Then $\mathcal{T}_\Sigma^{\leftarrow\top} \equiv_\Sigma^S \mathcal{T}$. $\mathcal{T}_\Sigma^{\leftarrow\top}$ is therefore a S_Σ-module of $\mathcal{T}$.*

We show by way of counter example that $\mathcal{T}_\Sigma^{\leftarrow\top}$ is not a self-contained or depleting S_Σ module of $\mathcal{T}$ when $\Sigma_\mathcal{T}^{\leftarrow\top} \neq Sig(\mathcal{T}_\Sigma^{\leftarrow\top})$.

Example 3. Let $\mathcal{T}$ be a TBox such that $\mathcal{T} = \{\alpha_1 = A \sqsubseteq \exists r.D_1, \alpha_2 = B \sqsubseteq \geq nr.D_2, \alpha_3 = \exists r.\top \sqsubseteq C, \alpha_4 = D_1 \sqsubseteq D_2\}$, and let $\Sigma = \{C\}$. Then $\mathcal{T}_\Sigma^{\leftarrow\top} = \{\alpha_1, \alpha_2, \alpha_3\}$, $\delta = \Sigma \cup Sig(\mathcal{T}_\Sigma^{\leftarrow\top}) = \{A, B, C, r, D_1, D_2\} \neq \Sigma_\mathcal{T}^{\leftarrow\top}$. But $\mathcal{T} \models D_1 \sqsubseteq D_2$ and $\mathcal{T}_\Sigma^{\leftarrow\top} \not\models D_1 \sqsubseteq D_2$. Therefore $\mathcal{T}_\Sigma^{\leftarrow\top}$ is not a self-contained c_Σ-module of $\mathcal{T}$. Similarly, $\mathcal{T} \setminus \mathcal{T}_\Sigma^{\leftarrow\top} \models \alpha_4 \neq \emptyset$ with $\Sigma = D_1, D_2$ and $D_1, D_2 \in \delta$. Therefore, $\mathcal{T}_\Sigma^{\leftarrow\top}$ is not a depleting c_Σ-module of $\mathcal{T}$.

Before investigating the robustness properties of reachability modules we introduce some lemmas to aid us in the proofs that follow.

Lemma 2. *Let α be an axiom, Σ and Σ' be signatures, $x \in \{\top, \bot\}$ and $\mathcal{T}$ a $\mathcal{SRIQ}$ TBox. Then:*

1. *If $\Sigma \subseteq \Sigma'$ and α is not ${\Sigma'}_{\mathcal{T}}^{\leftarrow x}$ reachable, then α is not $\Sigma_{\mathcal{T}}^{\leftarrow x}$ reachable.*
2. *If $\Sigma' \cap Sig(\alpha) \subseteq \Sigma$ and α is not Σ reachable then α is not Σ' reachable.*

Lemma 3. *Let α be an axiom, Σ and Σ' be signatures, $x \in \{\top, \bot\}$ and $\mathcal{T}$, $\mathcal{T}'$ $\mathcal{SRIQ}$ TBoxes. Then:*

1. *Given $\mathcal{T}_1 = \mathcal{T}_{\Sigma'}^{\leftarrow x}$, if $\Sigma \subseteq \Sigma'$ then $\mathcal{T}_{\Sigma}^{\leftarrow x} = {\mathcal{T}_1}_{\Sigma}^{\leftarrow x}$. In particular $\mathcal{T}_{\Sigma}^{\leftarrow x} \subseteq \mathcal{T}_{\Sigma'}^{\leftarrow x}$.*
2. *If $\Sigma' \cap Sig(\mathcal{T}) \subseteq \Sigma$, then $\mathcal{T}_{\Sigma'}^{\leftarrow x} \subseteq \mathcal{T}_{\Sigma}^{\leftarrow x}$.*
3. *If $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}_{\Sigma}^{\leftarrow x} \subseteq {\mathcal{T}'}_{\Sigma}^{\leftarrow x}$.*

Lemma 4. *Let Σ be an signature, $\mathcal{T}_1$ and $\mathcal{T}_2$ be $\mathcal{SRIQ}$ TBoxes with $Sig(\mathcal{T}_1) \cap Sig(\mathcal{T}_2) \subseteq \Sigma$ and $x \in \{\top, \bot\}$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)_{\Sigma}^{\leftarrow x} = {\mathcal{T}_1}_{\Sigma}^{\leftarrow x} \cup {\mathcal{T}_2}_{\Sigma}^{\leftarrow x}$.*

Proposition 1. *For $x \in \{\top, \bot\}$, x-reachability is robust under replacement.*

Proposition 2. *For $x \in \{\top, \bot\}$, x-reachability is robust under vocabulary extensions.*

Proposition 3. *For $x \in \{\top, \bot\}$, x-reachability is robust under vocabulary restrictions.*

Proposition 4. *For $x \in \{\top, \bot\}$, x-reachability is robust under joins.*

Reachability modules therefore share all the robustness properties listed. However, we have seen that these modules are neither depleting nor self-contained modules. Amongst other things, the depleting and self-contained nature of modules are utilised in order to find all MinAs (justifications) for an entailment [6].

Definition 12. *Let $\mathcal{T}$ be a $\mathcal{SRIQ}$ Tbox and $\mathcal{M} \subseteq \mathcal{T}$. $\mathcal{M}$ is a MinA (justification) for $\mathcal{T} \models C \sqsubseteq D$ if $\mathcal{M} \models C \sqsubseteq D$ and there exists no $\mathcal{M}_1 \subset \mathcal{M}$ such that $\mathcal{M}_1 \models C \sqsubseteq D$.*

We show that although our modules do not share these properties they do contain all MinAs for a given signature.

Theorem 2. *Let $\mathcal{T}$ be a normalized $\mathcal{SRIQ}$ TBox and Σ a signature such that $\Sigma \subseteq Sig(\mathcal{T})$. Then for arbitrary concept descriptions C, D, such that $\mathcal{T} \models C \sqsubseteq D$ and $Sig(C) \cup Sig(D) \subseteq \Sigma_{\mathcal{T}}^{\leftarrow \top}$ we have that $\mathcal{T}_{\Sigma}^{\leftarrow \top}$ contains all MinAs for $\mathcal{T} \models C \sqsubseteq D$.*

The proof to show that $\mathcal{T}_{\Sigma}^{\leftarrow \bot\top^*}$ modules share all the robustness properties of $\mathcal{T}_{\Sigma}^{\leftarrow \top}$ modules follows from the above lemmas and follows the proof for $\bot\top^*$-locality modules by Sattler, et al. [13].

5 Conclusion

We have investigated the module-theoretic properties of reachability modules for $\mathcal{SRIQ}$ TBoxes. Reachability modules differ from syntactic locality modules in that they are not self-contained or depleting. One application of the self-contained and depleting nature of locality modules is the finding of all justifications for entailments. However, in terms of finding justifications, by showing that reachability modules do preserve all justifications for entailments, we have shown that these properties are sufficient but that they are not necessary.

We did preliminary investigations into the size difference between locality and reachability modules. We extracted a random sample of 1000 modules from each of the Pizza, Nci, Nap and Xylocopa_v4 ontologies. Reachability modules were between 2.5% and 33% smaller than locality modules with an average of 22% reduction in size across all ontologies tested.

Our focus for future research is to do an in-depth empirical evaluation on differences with respect to size and performance between extracting reachability modules for $\mathcal{SRIQ}$ and existing syntactic locality methods. We also plan to extend these results to $\mathcal{SROIQ}$.

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A Proofs for Theorems and Lemmas

Theorem 1 Exhaustively applying the rules from Table 2 to any $\mathcal{SRIQ}$ TBox $\mathcal{T}$ results in a $\mathcal{SRIQ}$ TBox $\mathcal{T}'$ in normal form. The normalization process can be completed in linear time in the number of axioms.

Proof: We show that any $\mathcal{SRIQ}$ TBox can be converted to an equivalent normal form as follows:

- Step 1 - $\equiv$-elimination: Rule NR11 may be applied at most once for each axiom in the TBox. No other rule introduces new axioms that contain equivalences. Therefore the elimination of all equivalences from the TBox will require linear time and add at most a linear number of axioms.
- Step 2 - $\forall$-elimination: Applying rule NR7 to every occurrence of a universal restriction in any axiom, irrespective of order, will result in the elimination of all universal restrictions within that axiom. Nested restrictions are handled recursively as they are removed and inserted into the added axioms as $\hat{C}$. There are a constant number of universal restrictions per axiom and therefore the application of rule NR7 will run in constant time, with each application adding a constant number of new axioms. Therefore eliminating all universal restrictions across all axioms will require linear time and add at most a linear number of new axioms.
- Step 3 - $\leqslant$-elimination: Applying rule NR10 to every occurrence of an atmost restriction in any axiom, irrespective of order, will result in the elimination of all atmost restrictions within that axiom. Nested restrictions are handled recursively as needed. There are a constant number of atmost restrictions per axiom and therefore the application of rule NR10 will run in constant time. Therefore eliminating all atmost restrictions across all axioms will require linear time.
- Step 4 - Complex role-filler elimination: At the start of this step there are no universal or at most restrictions. We eliminate all complex role fillers by applying rules NR8 and NR9 to all axioms. There are a constant number of complex role fillers per axiom, therefore the application of these rules requires constant time per axiom and will add at most a constant number of axioms. Therefore, removing all complex role fillers from an ontology by using rules NR8 and NR9 will require at most linear time and add a linear number of new axioms.
- Step 5 - $\neg$-elimination and $\sqcap$, $\sqcup$-simplification: At this step there are no existential restriction or at-least restriction with complex role fillers. Given any axiom $\alpha = (\alpha_L \sqsubseteq \alpha_R)$, in a left to right fashion we apply the rules as follows:
  1. Apply rules NR1, NR3, NR6 to α_L until α_L consists of an existential restriction, an at-least restriction or the conjunction of concept names. There are a constant number of complex concept description, negations, disjunctions and conjunctions in α_L, each of these rules either eliminates a negation, removes a complex concept description or eliminates a disjunction. There are a constant number of times each of these operations

may be applied. Rules NR3 and NR6 each add a constant number of axioms. Therefore, α_L can be processed in constant time for each axiom.

2. Apply rules NR2, NR4, NR5 to α_R until α_R consists of an existential restriction, an at-least restriction or a disjunction of concept names. There are a constant number of times each of these operations may be applied. Rules NR4 and NR5 each add a constant number of axioms. Therefore, α_L can be processed in constant time for each axiom.

These rules are applied repeatedly until no further processing may be applied to either α_L and α_R. Since each step can be completed in constant time and add at most a constant number of new axioms, normalization can be completed in linear time in the number of axioms with at most a linear increase in the number of axioms. □

Lemma 1 Let $\mathcal{T}$ be a $\mathcal{SRIQ}$ TBox, and $\Sigma \subseteq \mathrm{Sig}(\mathcal{T})$ a signature. Then $\mathcal{T} \models C \sqsubseteq D$ if and only if $\mathcal{T}_{\Sigma}^{\leftarrow\top} \models C \sqsubseteq D$ for arbitrary $\mathcal{SRIQ}$ concept descriptions C and D such that $Sig(C) \cup Sig(D) \subseteq \Sigma$.

Proof: We have to prove two parts. First: If $\mathcal{T}_{\Sigma}^{\leftarrow\top} \models C \sqsubseteq D$ then $\mathcal{T} \models C \sqsubseteq D$. This follows directly from the fact that $\mathcal{T}_{\Sigma}^{\leftarrow\top} \subseteq \mathcal{T}$ and that $\mathcal{SRIQ}$ is monotonic.

Conversely, we show that, if $\mathcal{T} \models C \sqsubseteq D$ then $\mathcal{T}_{\Sigma}^{\leftarrow\top} \models C \sqsubseteq D$. Assume the contrary, that is, assume $\mathcal{T} \models C \sqsubseteq D$ but that $\mathcal{T}_{\Sigma}^{\leftarrow\top} \not\models C \sqsubseteq D$. Then there must exist an interpretation $\mathcal{I}$ and an individual $w \in \Delta^{\mathcal{I}}$ such that $\mathcal{I}$ is a model of $\mathcal{T}_{\Sigma}^{\leftarrow\top}$ and $w \in C^{\mathcal{I}} \setminus D^{\mathcal{I}}$. Modify $\mathcal{I}$ to $\mathcal{I}'$ by setting $x^{\mathcal{I}'} := \Delta^{\mathcal{I}}$ for all concept names $x \in Sig(\mathcal{T}) \setminus \Sigma_{\mathcal{T}}^{\leftarrow\top}$, and $r^{\mathcal{I}'} := \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for all roles names $r \in Sig(\mathcal{T}) \setminus \Sigma^{\leftarrow\top}$ and leaving everything else unchanged. We show that $\mathcal{I}'$ is a model of $\mathcal{T}_{\Sigma}^{\leftarrow\top}$. For all $\alpha := \alpha_L \sqsubseteq \alpha_R$, with $\alpha \in \mathcal{T}_{\Sigma}^{\leftarrow\top}$, we have that:

- If α_R is such that $Sig(\alpha_R) \subseteq \Sigma_{\mathcal{T}}^{\leftarrow\top}$ we have that $(\alpha_R)^{\mathcal{I}} = (\alpha_R)^{\mathcal{I}'}$ since it does not change the interpretation of any symbols.
- If α_R is an existential restriction of the form $\exists r.A$ with $y \in Sig(\alpha_R) \setminus \Sigma_{\mathcal{T}}^{\leftarrow\top}$, then $(y)^{\mathcal{I}'} = \Delta^{\mathcal{I}}$ or $(y)^{\mathcal{I}'} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ depending on whether y is a role or concept name. In both cases we have that $(\alpha_R)^{\mathcal{I}} \subseteq (\alpha_R)^{\mathcal{I}'}$.
- If α_R is an at-least restriction of the form $\geq nr.A$ with $y \in Sig(\alpha_R) \setminus \Sigma_{\mathcal{T}}^{\leftarrow\top}$, then $(y)^{\mathcal{I}'} = \Delta^{\mathcal{I}}$ or $(y)^{\mathcal{I}'} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ depending on whether y is a role or concept name. In both cases we have that $(\alpha_R)^{\mathcal{I}} \subseteq (\alpha_R)^{\mathcal{I}'}$.
- If α_R is of the form $\exists R.Self$ with $R \in \Sigma_{\mathcal{T}}^{\leftarrow\top}$ we have that $(\alpha_R)^{\mathcal{I}} = (\alpha_R)^{\mathcal{I}'}$ since it does not change the interpretation of the symbol R.
- If α is of the form $Dis(R, S)$ then by definition it is always in $\mathcal{T}_{\Sigma}^{\leftarrow\top}$, thus $R, S \in \Sigma_{\mathcal{T}}^{\leftarrow\top}$. Therefore, the interpretation of alpha does not change.

In all cases $(\alpha_L)^{\mathcal{I}} = (\alpha_L)^{\mathcal{I}'}$ since $\alpha \in \mathcal{T}_{\Sigma}^{\leftarrow\top}$ and $Sig(\alpha_L) \in \Sigma_{\mathcal{T}}^{\leftarrow\top}$ and thus $(\alpha_L)^{\mathcal{I}'} \subseteq (\alpha_R)^{\mathcal{I}'}$. Thus, $\mathcal{I}'$ is a model for $\mathcal{T}_{\Sigma}^{\leftarrow\top}$. Now for every $\alpha = (\alpha_L \sqsubseteq \alpha_R) \in \mathcal{T} \setminus \mathcal{T}_{\Sigma}^{\leftarrow\top}$ we have:

- α_R is a concept name and $\alpha_R^{\mathcal{I}'} = \Delta^{\mathcal{I}}$, or
- α_R is a role name and $\alpha_R^{\mathcal{I}'} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, or

– α_R is a disjunction of the form $A_1 \sqcup \ldots \sqcup A_n$ with at least one $A_i \notin \Sigma_{\mathcal{T}}^{\leftarrow\top}$, thus $A_i^{\mathcal{I}'} = \Delta^{\mathcal{I}}$ and $\alpha_R^{\mathcal{I}'} = A_1^{\mathcal{I}} \cup \ldots \cup \Delta^{\mathcal{I}} \cup \ldots \cup A_n^{\mathcal{I}} = \Delta^{\mathcal{I}}$, or
– α_R is an existential restriction $\exists r.A_1$, thus $r^{\mathcal{I}'} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and $A_1^{\mathcal{I}'} = \Delta^{\mathcal{I}}$ so that $(\exists r.A_1)^{\mathcal{I}'} = \Delta^{\mathcal{I}}$, or
– α_R is $\exists r.Self$, thus $r^{\mathcal{I}'} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ so that $(\exists r.Self)^{\mathcal{I}'} = \Delta^{\mathcal{I}}$, or
– α_R is an atleast restriction $\geq nr.A_2$, thus $r^{\mathcal{I}'} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $A_2^{\mathcal{I}'} = \Delta^{\mathcal{I}}$ and $|\Delta^{\mathcal{I}}| \geq n$ so that $(\geq nr.A_2)^{\mathcal{I}'} = \Delta^{\mathcal{I}}$. This follows from the fact that for any concept description $\geq nr.A$, $|\Delta^{\mathcal{I}}| \geq |(r.A)^{\mathcal{I}}| \geq n$ for it to be satisfiable.

Since for all cases $\alpha_L^{\mathcal{I}'} \subseteq \alpha_R^{\mathcal{I}'}$, we conclude that $\mathcal{I}'$ is a model for $\mathcal{T}$. But $\mathcal{I}$ and $\mathcal{I}'$ correspond on all symbols $y \in (Sig(D) \cup Sig(C)) \subseteq \Sigma \subseteq \Sigma_{\mathcal{T}}^{\leftarrow\top}$ and therefore $D^{\mathcal{I}'} = D^{\mathcal{I}}$ and $C^{\mathcal{I}'} = C^{\mathcal{I}}$. Now since $C^{\mathcal{I}} = C^{\mathcal{I}'}$ and $w \in C^{\mathcal{I}}$ we have that $w \in C^{\mathcal{I}'} \setminus D^{\mathcal{I}'}$ and hence $\mathcal{T} \not\models C \sqsubseteq D$, contradicting the assumption. □

Lemma 2 Let α be an axiom, Σ and Σ' be signatures, $x \in \{\top, \bot\}$ and $\mathcal{T}$ a $\mathcal{SRIQ}$ TBox. Then:

1. If $\Sigma \subseteq \Sigma'$ and α is not ${\Sigma'}_{\mathcal{T}}^{\leftarrow x}$ reachable, then α is not $\Sigma_{\mathcal{T}}^{\leftarrow x}$ reachable.
2. If $\Sigma' \cap Sig(\alpha) \subseteq \Sigma$ and α is not Σ reachable then α is not Σ' reachable.

Proof:

1. By the inductive definition of x-reachability if $\Sigma \subseteq \Sigma'$ then $\Sigma_{\mathcal{T}}^{\leftarrow x} \subseteq {\Sigma'}_{\mathcal{T}}^{\leftarrow x}$. Thus if α is not ${\Sigma'}_{\mathcal{T}}^{\leftarrow x}$ reachable it can also not be $\Sigma_{\mathcal{T}}^{\leftarrow x}$-reachable.
2. Assume that α is not Σ reachable but it is Σ' reachable. Then there is some symbol $y \in Sig(\alpha)$ such that $y \notin \Sigma$ and y is required for α to be Σ reachable. α is Σ' reachable so it must be the case that $y \in \Sigma'$. But this contradicts our assumption that $\Sigma' \cap Sig(\alpha) \subseteq \Sigma$. Thus, α is not Σ' reachable.

Lemma 3 Let α be an axiom, Σ and Σ' be signatures, $x \in \{\top, \bot\}$ and $\mathcal{T}$, $\mathcal{T}'$ $\mathcal{SRIQ}$ TBoxes. Then:

1. Given $\mathcal{T}_1 = \mathcal{T}_{\Sigma'}^{\leftarrow x}$, if $\Sigma \subseteq \Sigma'$ then $\mathcal{T}_{\Sigma}^{\leftarrow x} = {\mathcal{T}_1}_{\Sigma}^{\leftarrow x}$. In particular $\mathcal{T}_{\Sigma}^{\leftarrow x} \subseteq \mathcal{T}_{\Sigma'}^{\leftarrow x}$.
2. If $\Sigma' \cap Sig(\mathcal{T}) \subseteq \Sigma$, then $\mathcal{T}_{\Sigma'}^{\leftarrow x} \subseteq \mathcal{T}_{\Sigma}^{\leftarrow x}$.
3. If $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}_{\Sigma}^{\leftarrow x} \subseteq {\mathcal{T}'}_{\Sigma}^{\leftarrow x}$.

Proof:

1. Assume that there is some axiom $\alpha \in \mathcal{T}_{\Sigma}^{\leftarrow x}$ such that $\alpha \notin \mathcal{T}_{\Sigma'}^{\leftarrow x}$. Therefore, we have that α is not ${\Sigma'}_{\mathcal{T}}^{\leftarrow x}$ reachable but that it is $\Sigma_{\mathcal{T}}^{\leftarrow x}$ reachable. But this is a contradiction by Lemma 2.1 since $\Sigma \subseteq \Sigma'$. Thus, $\mathcal{T}_{\Sigma}^{\leftarrow x} \subseteq \mathcal{T}_{\Sigma'}^{\leftarrow x}$. A similar argument is used to show that $\mathcal{T}_{\Sigma}^{\leftarrow x} \subseteq {\mathcal{T}_1}_{\Sigma}^{\leftarrow x}$ and ${\mathcal{T}_1}_{\Sigma}^{\leftarrow x} \subseteq \mathcal{T}_{\Sigma}^{\leftarrow x}$.
2. For every $\alpha \subseteq \mathcal{T}$ we have that $\Sigma' \cap Sig(\alpha) \subseteq \Sigma$. Therefore, from Lemma 2.2 we have that whenever $\alpha \subseteq \mathcal{T}$ is not Σ reachable it is also not Σ' reachable and we have that $\mathcal{T}_{\Sigma'}^{\leftarrow x}$ contains at most all those axioms in $\mathcal{T}_{\Sigma}^{\leftarrow x}$. Thus, $\mathcal{T}_{\Sigma'}^{\leftarrow x} \subseteq \mathcal{T}_{\Sigma}^{\leftarrow x}$.
3. Let $\delta = \Sigma_{\mathcal{T}}^{\leftarrow x}$, $\delta' = {\Sigma'}_{\mathcal{T}_1}^{\leftarrow x}$ and $\alpha \in (\mathcal{T} \cap \mathcal{T}_1)$. Assume α is δ reachable but not δ' reachable. Since $\mathcal{T} \subseteq \mathcal{T}_1$ and $Sig(\mathcal{T}) \subseteq Sig(\mathcal{T}_1)$ we have by the inductive definition of x reachability that $\delta \subseteq \delta'$. But by Lemma 2.1 we have that whenever α is not δ' reachable then it is also not δ reachable. Therefore, $\mathcal{T}_{\Sigma}^{\leftarrow x}$ contains at most all those axioms in ${\mathcal{T}_1}_{\Sigma}^{\leftarrow x}$. Thus, $\mathcal{T}_{\Sigma}^{\leftarrow x} \subseteq {\mathcal{T}_1}_{\Sigma}^{\leftarrow x}$.

Lemma 4 Let Σ be an signature, $\mathcal{T}_1$ and $\mathcal{T}_2$ be $\mathcal{SRIQ}$ TBoxes with $Sig(\mathcal{T}_1) \cap Sig(\mathcal{T}_2) \subseteq \Sigma$ and $x \in \{\top, \bot\}$. Then $(\mathcal{T}_1 \cup \mathcal{T}_2)^{\leftarrow x}_{\Sigma} = \mathcal{T}_1{}^{\leftarrow x}_{\Sigma} \cup \mathcal{T}_2{}^{\leftarrow x}_{\Sigma}$.

Proof: Let $\mathcal{M} = (\mathcal{T}_1 \cup \mathcal{T}_2)^{\leftarrow x}_{\Sigma}$, $\mathcal{M}_1 = \mathcal{T}_1{}^{\leftarrow x}_{\Sigma}$, $\mathcal{M}_2 = \mathcal{T}_2{}^{\leftarrow x}_{\Sigma}$. Now $\mathcal{T}_1 \subseteq \mathcal{T}_1 \cup \mathcal{T}_2$ thus by Lemma 3.3 we have that $\mathcal{M}_1 \subseteq \mathcal{M}$. Similarly $\mathcal{M}_2 \subseteq \mathcal{M}$ and thus $\mathcal{M}_1 \cup \mathcal{M}_2 \subseteq \mathcal{M} \cup \mathcal{M}$ which gives us $\mathcal{M}_1 \cup \mathcal{M}_2 \subseteq \mathcal{M}$. Let $\Sigma' = \Sigma \cup \Sigma^{\leftarrow x}_{\mathcal{T}_1} \cup \Sigma^{\leftarrow x}_{\mathcal{T}_2}$. To show that $\mathcal{M} \subseteq \mathcal{M}_1 \cup \mathcal{M}_2$ we note that, when extracting these modules, the order in which axioms are extracted are irrelevant. We therefore assume that any algorithm first extracts axioms in $\mathcal{M}_1 \cup \mathcal{M}_2$ then tests all other axioms for $\Sigma'^{\leftarrow x}_{\mathcal{T}_1 \cup \mathcal{T}_2}$-reachability. Consider any axiom $\alpha \in (\mathcal{T}_1 \cup \mathcal{T}_2) \setminus (\mathcal{M}_1 \cup \mathcal{M}_2)$. If $\alpha \in \mathcal{T}_1$ then $\alpha \in \mathcal{T}_1 \setminus \mathcal{M}_1$ and α is not $\Sigma^{\leftarrow x}_{\mathcal{T}_1} \cup \Sigma$ reachable. Now precondition $Sig(\mathcal{T}_2) \cap Sig(\mathcal{T}_1) \subseteq \Sigma$ implies $\Sigma^{\leftarrow x}_{\mathcal{T}_2} \cap Sig(\alpha) \subseteq \Sigma$, taken that α is not $\Sigma^{\leftarrow x}_{\mathcal{T}_1} \cup \Sigma$ reachable we manipulate this statement to derive $(\Sigma \cup \Sigma^{\leftarrow x}_{\mathcal{T}_2} \cup \Sigma^{\leftarrow x}_{\mathcal{T}_1}) \cap Sig(\alpha) \subseteq \Sigma \cup \Sigma^{\leftarrow x}_{\mathcal{T}_1}$. Thus by Lemma 2.2 we have that α is not $\Sigma \cup \Sigma^{\leftarrow x}_{\mathcal{T}_2} \cup \Sigma^{\leftarrow x}_{\mathcal{T}_1}$ reachable. The case where $\alpha \in \mathcal{T}_2$ is treated analogously. □

Proposition 1 For $x \in \{\top, \bot\}$, x-reachability is robust under replacement.

Proof: Let $Sig(\mathcal{T}) \cap (Sig(\mathcal{T}_1) \cup Sig(\mathcal{T}_2)) \subseteq \Sigma$. This implies that $Sig(\mathcal{T}) \cap Sig(\mathcal{T}_i) \subseteq \Sigma$, for $i = 1, 2$. Now we have:

$$\begin{aligned}(\mathcal{T} \cup \mathcal{T}_1)^{\leftarrow x}_{\Sigma} &= \mathcal{T}^{\leftarrow x}_{\Sigma} \cup \mathcal{T}_1{}^{\leftarrow x}_{\Sigma} \text{ (Lemma 4)}\\ &= \mathcal{T}_2{}^{\leftarrow x}_{\Sigma} \cup \mathcal{T}^{\leftarrow x}_{\Sigma} \text{ (Precondition)}\\ &= (\mathcal{T}_2 \cup \mathcal{T})^{\leftarrow x}_{\Sigma} \quad \text{(Lemma 4)} \quad \square\end{aligned}$$

Proposition 2 For $x \in \{\top, \bot\}$, x-reachability is robust under vocabulary extensions.

Proof: Let $\Sigma' \cap (Sig(\mathcal{T}_1) \cup Sig(\mathcal{T}_2)) \subseteq \Sigma$ and $\mathcal{T}_1 \equiv^x_{\Sigma} \mathcal{T}_2$ i.e., $\mathcal{T}_1{}^{\leftarrow x}_{\Sigma} = \mathcal{T}_2{}^{\leftarrow x}_{\Sigma}$. Let $\Sigma'' = \Sigma \cap \Sigma'$ (which implies $\Sigma'' \subseteq \Sigma$ and $\Sigma'' \subseteq \Sigma'$). Then we have that:

$$\begin{aligned}\mathcal{T}_1{}^{\leftarrow x}_{\Sigma'} &= \mathcal{T}_1{}^{\leftarrow x}_{\Sigma''} (*)\\ &= (\mathcal{T}_1{}^{\leftarrow x}_{\Sigma})^{\leftarrow x}_{\Sigma''} \text{ (Lemma 3.1)}\\ &= (\mathcal{T}_2{}^{\leftarrow x}_{\Sigma})^{\leftarrow x}_{\Sigma''} \text{ (Precondition)}\\ &= \mathcal{T}_2{}^{\leftarrow x}_{\Sigma''} \qquad \text{(Lemma 3.1)}\\ &= \mathcal{T}_2{}^{\leftarrow x}_{\Sigma'} (**)\end{aligned}$$

As for equality $(*)$, set inclusion is due to $\Sigma' \cap Sig(\mathcal{T}_1) = \Sigma''$ and the combination of Lemma 3.2 and Lemma 3.3, and the converse is due to $\Sigma'' \subseteq \Sigma'$ and Lemma 3.1. Equality $(**)$ is justified analogously. □

Proposition 3 For $x \in \{\top, \bot\}$, x-reachability is robust under vocabulary restrictions.

Proof: Follows from the fact that we have robustness under vocabulary extensions. □

Proposition 4 For $x \in \{\top, \bot\}$, x-reachability is robust under joins.

Proof: For $i = 1, 2$, let $\mathcal{M}_i = \mathcal{T}_{i\,\Sigma}^{\leftarrow x}$ with $Sig(\mathcal{T}_1) \cap Sig(\mathcal{T}_2) \subseteq \Sigma$, and let $\mathcal{M} = (\mathcal{T}_1 \cup \mathcal{T}_2)_{\Sigma}^{\leftarrow x}$. The precondition says that $\mathcal{M}_1 = \mathcal{M}_2$. It is clear from Lemma 3.3 that $\mathcal{M} \supseteq \mathcal{M}_i$. It suffices to show $\mathcal{M} \subseteq \mathcal{M}_1$. Take any axiom $\alpha \in (\mathcal{T}_1 \cup \mathcal{T}_2) \setminus \mathcal{M}_1$. It remains to show that α is not $\Sigma \cup \Sigma_{\mathcal{M}_1}^{\leftarrow x}$ reachable. In case $\alpha \in \mathcal{T}_1 \setminus \mathcal{M}_1$, then α is not $\Sigma_{\mathcal{M}_1}^{\leftarrow x}$ reachable since $\mathcal{M}_1 = \mathcal{T}_{1\,\Sigma}^{\leftarrow x}$. In case $\alpha \in \mathcal{T}_2 \setminus \mathcal{M}_1$, we also have that $\alpha \in \mathcal{T}_2 \setminus \mathcal{M}_2$ because $\mathcal{M}_1 = \mathcal{M}_2$. This means that α is not $\Sigma \cup \Sigma_{\mathcal{M}_2}^{\leftarrow x}$ reachable therefore not $\Sigma \cup \Sigma_{\mathcal{M}_1}^{\leftarrow x}$ reachable. □

Theorem 2 Let $\mathcal{T}$ be a normalized $\mathcal{SRIQ}$ TBox and Σ a signature such that $\Sigma \subseteq Sig(\mathcal{T})$. For arbitrary concept descriptions C, D such that $\mathcal{T} \models C \sqsubseteq D$ and $Sig(C) \cup Sig(D) \subseteq \Sigma_{\mathcal{T}}^{\leftarrow \top}$ we have that $\mathcal{T}_{\Sigma}^{\leftarrow \top}$ contains all MinAs for $\mathcal{T} \models C \sqsubseteq D$.

Proof: Assume that $\mathcal{T} \models C \sqsubseteq D$ for some $Sig(C) \cup Sig(D) \subseteq \Sigma_{\mathcal{T}}^{\leftarrow \top}$, but there is a MinA $\mathcal{M}$ for $\mathcal{T} \models C \sqsubseteq D$ that is not contained in $\mathcal{T}_{\Sigma}^{\leftarrow \top}$. If $C \sqsubseteq D$ is a tautology then $\mathcal{M}$ must be empty with $\mathcal{M} \subseteq \mathcal{T}_{\Sigma}^{\leftarrow \top}$. Thus, we assume that $C \sqsubseteq D$ is not a tautology. Since $\mathcal{M} \not\subseteq \mathcal{T}_{\Sigma}^{\leftarrow \top}$, there must be an axiom $\alpha \in \mathcal{M} \setminus \mathcal{T}_{\Sigma}^{\leftarrow \top}$. Define $\mathcal{M}_1 := \mathcal{M} \cap \mathcal{T}_{\Sigma}^{\leftarrow \top}$. $\mathcal{M}_1$ is a strict subset of $\mathcal{M}$ since $\alpha \notin \mathcal{M}_1$. There are two cases, either $\mathcal{M}_1 = \emptyset$ or it contains at least one axiom.

In the case where $\mathcal{M}_1 = \emptyset$, define $\mathcal{T}_1 = \mathcal{T} \setminus \mathcal{T}_{\Sigma}^{\leftarrow \top}$ with $\mathcal{M} \subseteq \mathcal{T}_1$. Now since $\mathcal{M} \models C \sqsubseteq D$ we have by monotinocity that $\mathcal{T}_1 \models C \sqsubseteq D$. Since $\mathcal{T}_1 \subseteq \mathcal{T}$ we have by Lemma 3.3 that $\mathcal{T}_{1\,\Sigma}^{\leftarrow \top} \subseteq \mathcal{T}_{\Sigma}^{\leftarrow \top}$ and thus that $\mathcal{T}_{1\,\Sigma}^{\leftarrow \top} = \emptyset$. But by Lemma 1 we have that $\mathcal{T}_{1\,\Sigma}^{\leftarrow \top} \models C \sqsubseteq D$ if, and only if, $\mathcal{T}_1 \models C \sqsubseteq D$. Since $C \sqsubseteq D$ is not a tautology we have that $\mathcal{T}_{1\,\Sigma}^{\leftarrow \top} \not\models C \sqsubseteq D$ and thus that $\mathcal{M} \not\models C \sqsubseteq D$.

In the case where $\mathcal{M}_1 \neq \emptyset$ we claim that $\mathcal{M}_1 \models C \sqsubseteq D$, which contradicts the fact that $\mathcal{M}$ is a MinA for $\mathcal{T} \models C \sqsubseteq D$.

We use proof by contraposition to show this. Assume that $\mathcal{M}_1 \not\models C \sqsubseteq D$, i.e., there is a model $\mathcal{I}_1$ of $\mathcal{M}_1$ such that $C^{\mathcal{I}_1} \not\subseteq D^{\mathcal{I}_1}$. We modify $\mathcal{I}_1$ to $\mathcal{I}$ by setting $y^{\mathcal{I}} := \Delta^{\mathcal{I}_1}$ for all concept names $y \in Sig(\mathcal{T}) \setminus \Sigma_{\mathcal{T}}^{\leftarrow \top}$, and $r^{\mathcal{I}} := \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_1}$ for all roles names $r \in Sig(\mathcal{T}) \setminus \Sigma_{\mathcal{T}}^{\leftarrow \top}$. We have $D^{\mathcal{I}} = D^{\mathcal{I}_1}$ since $Sig(D) \subseteq \Sigma_{\mathcal{T}}^{\leftarrow \top}$, and $C^{\mathcal{I}} = C^{\mathcal{I}_1}$ since $Sig(C) \subseteq \Sigma_{\mathcal{T}}^{\leftarrow \top}$. It follows that $C^{\mathcal{I}} \not\subseteq D^{\mathcal{I}}$. It remains to be shown that $\mathcal{I}$ is indeed a model of $\mathcal{M}$, and therefore satisfies all axioms $\beta = (\beta_L \sqsubseteq \beta_R)$ in $\mathcal{M}$, including α. If $\beta = Dis(R_r, R_2)$ then by definition $Sig(\beta) \subseteq \Sigma_{\mathcal{T}}^{\leftarrow \top}$ so that $(\beta)^{\mathcal{I}} = (\beta)^{\mathcal{I}^1}$. Otherwise there are two possibilities:

- $\beta \in \mathcal{M}_1$. Since $\mathcal{M}_1 \subseteq \mathcal{T}_{\Sigma}^{\leftarrow \top}$, all symbols in $Sig(\beta_L)$ are also in $\Sigma_{\mathcal{T}}^{\leftarrow \top}$ and possibly some symbols of Sig(β_R) may not be in $\Sigma_{\mathcal{T}}^{\leftarrow \top}$. Consequently, $\mathcal{I}_1$ and $\mathcal{I}$ coincide on the names occurring in β_L and since $\mathcal{I}_1$ is a model of $\mathcal{M}_1$, we have that $(\beta_L)^{\mathcal{I}} = (\beta_L)^{\mathcal{I}_1}$ and $(\beta_R)^{\mathcal{I}_1} \subseteq (\beta_R)^{\mathcal{I}}$. Therefore $(\beta_L)^{\mathcal{I}} \subseteq (\beta_R)^{\mathcal{I}}$.
- $\beta \notin \mathcal{M}_1$. Since $\beta \in \mathcal{M}$, we have that $\beta \notin \mathcal{T}_{\Sigma}^{\leftarrow \top}$, and hence β is not $\Sigma_{\mathcal{T}}^{\leftarrow \top}$-reachable. Thus,
  - β_R is a concept name and $\beta_R^{\mathcal{I}'} = \Delta^{\mathcal{I}}$, or
  - β_R is a role name and $\beta_R^{\mathcal{I}'} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, or
  - β_R is a disjunction of the form $A_1 \sqcup \ldots \sqcup A_n$ with at least one $A_i \notin \Sigma_{\mathcal{T}}^{\leftarrow \top}$, thus $A_i^{\mathcal{I}'} = \Delta^{\mathcal{I}}$ and $\beta_R^{\mathcal{I}'} = A_1^{\mathcal{I}} \cup \ldots \cup \Delta^{\mathcal{I}} \cup \ldots \cup A_n^{\mathcal{I}} = \Delta^{\mathcal{I}}$, or

- β_R is an existential restriction $\exists r.A_1$, thus $r^{\mathcal{I}'} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and $A_1^{\mathcal{I}'} = \Delta^{\mathcal{I}}$ so that $(\exists r.A_1)^{\mathcal{I}'} = \Delta^{\mathcal{I}}$, or
- β_R is $\exists r.Self$, thus $r^{\mathcal{I}'} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ so that $(\exists r.Self)^{\mathcal{I}'} = \Delta^{\mathcal{I}}$, or
- β_R is an atleast restriction $\geq nr.A_2$, thus $r^{\mathcal{I}'} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $A_2^{\mathcal{I}'} = \Delta^{\mathcal{I}}$ and $|\Delta^{\mathcal{I}}| \geq n$ so that $(\geq nr.A_2)^{\mathcal{I}'} = \Delta^{\mathcal{I}}$. This follows from the fact that for any concept description $\geq nr.A$, $|\Delta^{\mathcal{I}}| \geq |(r.A)^{\mathcal{I}}| \geq n$ for it to be satisfiable.

By definition of $\mathcal{I}$, $(\beta_R)^{\mathcal{I}} = \Delta^{\mathcal{I}_1}$. Hence $(\beta_L)^{\mathcal{I}} \subseteq (\beta_R)^{\mathcal{I}}$.

Therefore $\mathcal{I}$ is a model for $\mathcal{M}$. But since $C^{\mathcal{I}} \not\subseteq D^{\mathcal{I}}$ we have that $\mathcal{M} \not\models C \sqsubseteq D$ proving the contrapositive. $\square$