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VARIATIONAL FORMULATION OF THE METHOD OF LINES AND ITS APPLICATION TO THE WAVE PROPAGATION PROBLEMS

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Many of wave propagation problems describing by hyperbolic equations could be formulated in terms of the variational principles. In the paper we demonstrate how to derive the systems of ordinary differential equations of the method of lines directly from the Lagrangians of the corresponding variational formulations. The discussed method has several advantages in comparison with the traditional methods of deriving of the initial problems from the initial-boundary problems for partial differential equations. First, in Lagrangians we use a finite difference representation of the spatial derivatives of lower order than in equations. For example, in the wave equations we need to represent the second partial derivative of displacements by its finite difference but in the corresponding Lagrangian we use the finite difference representation of the first partial derivative. In the equations of longitudinal vibration of the Rayleigh-Bishop bar as well as in the equations of lateral vibration of the Euler-Bernoulli beam we need to use a finite difference representation of the fourth order partial derivatives of displacements, but in the Lagrangians we need a finite difference representation of the second order partial derivatives. The second advantage of the variational approach to the method of lines is connected with the fact that number of terms in Lagrangians is less than in the corresponding equations. For example, in the equation of vibration of the Rayleigh-Bishop bar with variable parameters there are eight terms including spatial partial derivatives of displacement of the first, second, third and fourth order and first and second partial derivatives of geometrical and physical parameters. In the corresponding Lagrangian there are four terms including first and second spatial partial derivatives of displacement, derivatives of combinations of geometrical and physical parameters are absent. Some limitations of the method are discussed in the paper.

1. Introduction

The method of lines is a simple and reliable method of numerical analysis of parabolic and hyperbolic problems of mathematical physics [1,2]. By means of this method mixed initial-boundary problems described by partial differential equations are transformed into systems of ordinary differential equations with initial conditions. This reduction is obtained by means of application of particular finite difference schemes to the spatial derivatives. Many of the wave propagation problems describing by the hyperbolic equations could be formulated in terms of the variational principles. In the present paper we demonstrate how to derive the systems of ordinary differential equations of the method of lines directly from the Lagrangians of the corresponding variational formulations of the wave propagation problems. The discussed method has several obvious advantages in comparison with the traditional methods of deriving of the initial problems of systems for ordinary differential equations from the initial-boundary problems for partial differential equations. First, in Lagrangians we need to use a finite difference representation of the spatial derivatives of lower order than in equations. For example, in the wave equations describing longitudinal vibration of bars, torsional vibration of rods, etc., we need to represent the second partial derivative of displacements by its finite difference but in the corresponding Lagrangian we need to use the finite difference representation of the first partial derivative of displacements. In the equations of longitudinal vibration of the Rayleigh-Bishop bar as well as in the equations of lateral vibration of the Euler-Bernoulli beam we need to use a finite difference representation of the fourth order partial derivatives of displacements, but in the corresponding Lagrangians we need a finite difference representation of the second order partial derivatives of displacements. The second advantage of the variational approach to the method of lines is connected with number of terms in equations and the corresponding Lagrangians. As a rule, number of terms in Lagrangians is substantially less than in the corresponding equations. For example, in the equation of vibration of the Rayleigh-Bishop bar with variable parameters there are eight terms including spatial partial derivatives of displacement of the first, second, third and fourth order and first and second partial derivatives of combinations of geometrical and physical parameters. In the corresponding Lagrangian there are four terms including first and second spatial partial derivatives of displacement and derivatives of combinations of geometrical and physical parameters are absent. Despite the obvious advantages of the variational formulation of the method of lines there are some limitations of its practical application which are also discussed in the paper.

2. Standard formulation by the method of lines

We have the following finite difference representations of first order derivative:

$$\left.\frac{\partial u(t,x)}{\partial x}\right|_{x=x_m} \approx \frac{1}{h}\left[u(t,x_{m+1})-u(t,x_m)\right]=\frac{1}{h}\left[u_{m+1}(t)-u_m(t)\right] \tag{1}$$

and

$$\left.\frac{\partial u(t,x)}{\partial x}\right|_{x=x_m} \approx \frac{1}{2h}\left[u(t,x_{m+1})-u(t,x_{m-1})\right]=\frac{1}{2h}\left[u_{m+1}(t)-u_{m+1}(t)\right] \tag{2}$$

where $h=x_{m+1}-x_m=x_m-x_{m-1}$, $m=0,1,2,\dots,N,N+1$.

Representation (2) is more accurate in comparison with (1) because its accuracy is $O\left[h^3\cdot u'''(t,x)\right]$ (for (1) the error has order $O\left[h^2\cdot u''(t,x)\right]$).

For the second order derivative:

$$\left.\frac{\partial^2 u(t,x)}{\partial x^2}\right|_{x=x_m} \approx \frac{1}{h^2}\left[u(t,x_{m-1})-2u(t,x_m)+u(t,x_{m+1})\right] \\ =\frac{1}{h^2}\left[u_{m-1}(t)-2u_m(t)+u_{m+1}(t)\right] \tag{3}$$

Estimation of error of this finite difference representation has order $O\left[h^4 \cdot u^{IV}(t,x)\right]$).

Let us consider the wave equation and show how to realize the method of lines. The wave equation describing longitudinal vibrations $u=u(t,x)$ of a bar of constant cross-section is as follows:

$$\rho\frac{\partial^2 u(t,x)}{\partial t^2}-E\frac{\partial^2 u(t,x)}{\partial x^2}=F(t,x) \tag{4}$$

where ρ is mass density and E is modulus of elasticity of the bar, $F(t,x)$ is the exciting force.

Let us assume that the bar is fixed at both left and right ends $u(t,x_0=0)=u(t,x_{N+1}=l)=0$. Its initial conditions are $u(t=0,x)=g(x)$ and $\left.\frac{du(t,x)}{dt}\right|_{t=0}=h(x)$. Hence, it is necessary to define motion of the rod in points $x=x_k$, $(k=1,2,\ldots,N)$, i.e. find $u_k(t)=u(t,x=x_k)$. To do this we use the finite difference approximation of the second spatial derivatives (3):

$$\left.\frac{\partial^2 u(t,x)}{\partial x^2}\right|_{x=x_k} \approx \frac{1}{h^2}\left[u_{k-1}(t)-2u_k(t)+u_{k+1}(t)\right] \tag{5}$$

and obtain the following system of ordinary differential equations:

$$\frac{d^2 u_k(t)}{dt^2}-\frac{c^2}{h^2}\left[u_{k-1}(t)-2u_k(t)+u_{k+1}(t)\right]=f_k(t) \tag{6}$$

where $c=\sqrt{\frac{E}{\rho}}$ is the speed of propagation of elastic wave over the bar and $f_k(t)=\frac{1}{\rho}F(t,x=x_k)$. If we add initial conditions $u_k(t=0)=g_k$ and $\left.\frac{du_k(t)}{dt}\right|_{t=0}=h_k$ to the system of ordinary differential equations (3.5) the initial problem will be formulated and could be solved by one of the available numerical methods (Runge-Kutta, Adams, etc.).

Explicit form of the system (3.5) is as follows:

$$\begin{gathered}\frac{d^2 u_1(t)}{dt^2}-\frac{c^2}{h^2}\left[-2u_1(t)+u_2(t)\right]=f_1(t), \\ \frac{d^2 u_2(t)}{dt^2}-\frac{c^2}{h^2}\left[u_1(t)-2u_2(t)+u_3(t)\right]=f_2(t), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \frac{d^2 u_{N-1}(t)}{dt^2}-\frac{c^2}{h^2}\left[u_{N-2}(t)-2u_{N-1}(t)+u_N(t)\right]=f_{N-1}(t),\end{gathered} \tag{7}$$

$$\frac{d^2u_N(t)}{dt^2}-\frac{c^2}{h^2}\left[u_{N-1}(t)-2u_N(t)\right]=f_N(t).$$

This system of equations with initial conditions: $u_k(t=0)=g_k$, $\left.\frac{du_k(t)}{dt}\right|_{t=0}=h_k$, $(k=1,2,\ldots,N)$ could be simply programmed and solved.

3. Variational formulation of the method of lines

The described method represents the classical approach to the method of line. It is based on discretization of spatial derivatives of a partial differential equation and obtaining a system of ordinary differential equations. Let us describe an alternative method, the so-called method of lines based on the variational approach. We know that the corresponding Lagrangian of the abovementioned problem is:

$$L=L(t)=\int_0^l\Lambda(t,x)\,dx=\int_0^l\frac{1}{2}\left\{\rho A\left[\frac{\partial u(t,x)}{\partial t}\right]^2-EA\left[\frac{\partial u(t,x)}{\partial x}\right]^2+AF(t,x)\cdot u(t,x)\right\}dx \qquad (8)$$

Equation (4) is obtained from Lagrangian (8) by means of the following Euler-Lagrange equation:

$$\frac{\partial}{\partial t}\left[\frac{\partial\Lambda(t,x)}{\partial\dot{u}}\right]+\frac{\partial}{\partial x}\left[\frac{\partial\Lambda(t,x)}{\partial u'}\right]-\frac{\partial\Lambda(t,x)}{\partial u}=0 \qquad (9)$$

Let us first make finite difference discretization (3.1a) of the first partial derivative in the expression for the Lagrangian density:

$$\begin{aligned}\Lambda(t,x=x_m)&=\frac{A}{2}\left\{\rho\left[\frac{du(t,x=x_m)}{dt}\right]^2-E\left[\frac{du(t,x=x_m)}{dx}\right]^2+F(t,x=x_m)\cdot u(t,x=x_m)\right\}\\&\approx\frac{A}{2}\left\{\rho\left[\dot{u}_m(t)\right]^2-\frac{E}{h^2}\left[u_{m+1}(t)-u_m(t)\right]^2+F_m(t)\cdot u_m(t)\right\}\approx\Lambda_m(t)\end{aligned} \qquad (10)$$

Using, as before, the fixed ends boundary conditions ($u_0(t)=u_{N+1}(t)=0$) we obtain the following explicit expressions for the partial Lagrangians:

$$\begin{aligned}\Lambda_0(t)&\approx-\frac{EA}{2h^2}\left[u_1(t)\right]^2,\\\Lambda_1(t)&\approx\frac{A}{2}\left\{\rho\left[\dot{u}_1(t)\right]^2-\frac{E}{h^2}\left[u_2(t)-u_1(t)\right]^2+F_1(t)\cdot u_1(t)\right\},\\&\ldots\ldots\ldots\ldots\ldots\ldots\ ,\\\Lambda_k(t)&\approx\frac{A}{2}\left\{\rho\left[\dot{u}_k(t)\right]^2-\frac{E}{h^2}\left[u_{k+1}(t)-u_k(t)\right]^2+F_k(t)\cdot u_k(t)\right\},\\&\ldots\ldots\ldots\ldots\ldots\ldots\ ,\\\Lambda_{N-1}(t)&\approx\frac{A}{2}\left\{\rho\left[\dot{u}_{N-1}(t)\right]^2-\frac{E}{h^2}\left[u_N(t)-u_{N-1}(t)\right]^2+F_{N-1}(t)\cdot u_{N-1}(t)\right\},\\\Lambda_N(t)&\approx\frac{A}{2}\left\{\rho\left[\dot{u}_N(t)\right]^2-\frac{E}{h^2}\left[u_N(t)\right]^2+F_N(t)\cdot u_N(t)\right\},\\\Lambda_{N+1}(t)&\approx\frac{A}{2}\left\{-\frac{E}{h^2}\left[u_N(t)\right]^2\right\}.\end{aligned} \qquad (11)$$

Using a simplest quadrature formula, namely the rectangle rule, for numerical calculation of integral (8) we obtain approximately:

$$L = L(t) = \int_0^l \Lambda(t,x)\,dx = h\cdot\left[\Lambda_0(t)+\Lambda_1(t)+\ldots\Lambda_N(t)\right]+O\left[h^2\cdot\frac{\partial\Lambda(t,x)}{\partial x}\right] \quad (12)$$

Substituting (11) in (12) we obtain the following approximate Lagrangian:

$$\begin{gathered} L \approx \frac{\rho A h}{2}\left(\dot{u}_1^2+\dot{u}_2^2+\ldots\dot{u}_N^2\right)-\frac{EA}{2h}\left[u_1^2+\left(u_2-u_1\right)^2+\ldots\left(u_N-u_{N-1}\right)^2+u_N^2\right] \\ +A\cdot h\cdot\left(F_1\cdot u_1+F_2\cdot u_2+\ldots F_N\cdot u_N\right)=L\left(\dot{u}_1,\dot{u}_2,\ldots\dot{u}_N;u_1,u_2,\ldots u_N\right) \end{gathered} \quad (13)$$

Hence, equations of this approximate model are given by the following Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_k}\right)-\frac{\partial L}{\partial u_k}=0, \qquad (k=1,2,\ldots,N) \quad (14)$$

For example,

$$\begin{aligned} k=1: &\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_1}\right)-\frac{\partial L}{\partial u_1}=\rho A h\ddot{u}_1-\frac{EA}{h}\left(u_2-2u_1\right)-hAF_1=0, \\ k=2: &\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_2}\right)-\frac{\partial L}{\partial u_2}=\rho A h\ddot{u}_2-\frac{EA}{h}\left(u_3-2u_2+u_1\right)-hAF_2=0, \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ k=N-1: &\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_{N-1}}\right)-\frac{\partial L}{\partial u_{N-1}}=\rho A h\ddot{u}_{N-1}-\frac{EA}{h}\left(u_N-2u_{N-1}+u_{N-2}\right)-hAF_{N-1}=0, \\ k=N: &\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_N}\right)-\frac{\partial L}{\partial u_N}=\rho A h\ddot{u}_N-\frac{EA}{h}\left(-2u_N+u_{N-1}\right)-hAF_N=0. \end{aligned} \quad (15)$$

It is obvious that systems of equations (15) and (7) are identical.

4. Lumped mass-spring representation

System (15) give us an opportunity to show an equivalent lumped mass representation of problem (4). It is represented in Fig. 1. In this figure we assume that $M=\rho A h$ is mass of the lumped mass element, $K=\frac{EA}{h}$ is stiffness of the lumped spring element, $\tilde{F}_k(t)=AhF_k(t)$ are the discrete analogs of the distributed exciting force and $u_k(t)$ are the displacements of the corresponding lumped masses.Let us proof that the lumped parameters mechanical system shown in Fig. 1 is described by system of equations (15). To do this we write the kinetic, potential energies, work of external forces and the Lagrangian of the lumped system (Fig. 1):

Kinetic energy of the system is:

$$K(t)=\frac{M}{2}\left\{\left[\dot{u}_1(t)\right]^2+\left[\dot{u}_2(t)\right]^2+\ldots+\left[\dot{u}_N(t)\right]^2\right\} \quad (16)$$

Strain energy of the system is:

$$P(t)=\frac{K}{2}\left\{\left[u_1(t)\right]^2+\left[u_2(t)-u_1(t)\right]^2+\ldots+\left[u_N(t)-u_{N-1}(t)\right]^2+\left[u_N(t)\right]^2\right\} \quad (17)$$

Work of external forces is:

$$W(t)=\tilde{F}_1(t)\cdot u_1(t)+\tilde{F}_2(t)\cdot u_2(t)+\ldots+\tilde{F}_N(t)\cdot u_N(t) \quad (18)$$

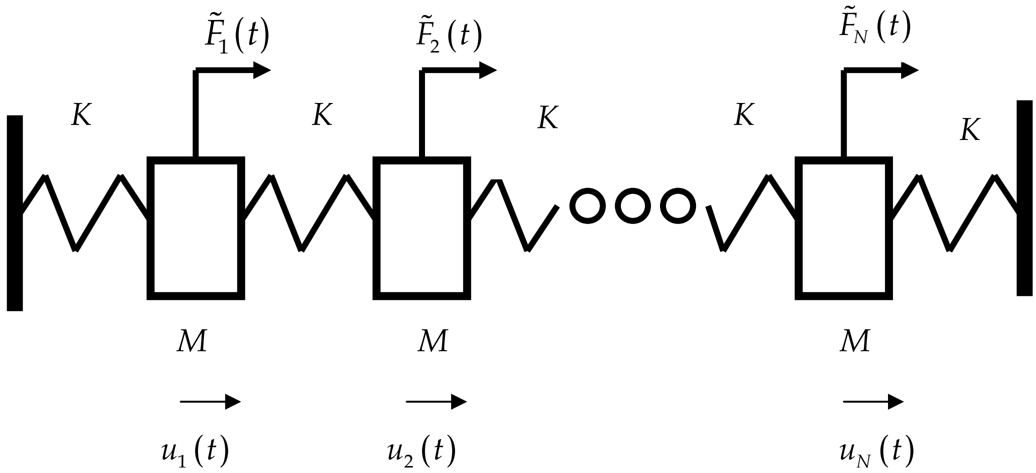

Figure 1. Equivalent lumped mass-spring representation of the distributed system

Lagrangian of the system is:

$$L(t)=K(t)-P(t)+W(t)$$
$$=\frac{1}{2}\left\{\begin{array}{l} M\left\{\left[\dot{u}_1(t)\right]^2+\left[\dot{u}_2(t)\right]^2+\ldots+\left[\dot{u}_N(t)\right]^2\right\} \\ -K\left\{\left[u_1(t)\right]^2+\left[u_2(t)-u_1(t)\right]^2+\ldots+\left[u_N(t)-u_{N-1}(t)\right]^2+\left[u_N(t)\right]^2\right\} \\ +2\left\{\tilde{F}_1(t)\cdot u_1(t)+\tilde{F}_2(t)\cdot u_2(t)+\ldots+\tilde{F}_N(t)\cdot u_N(t)\right\} \end{array}\right\} \quad (19)$$

Keeping in mind that $M=\rho Ah$, $K=\frac{EA}{h}$ and $\tilde{F}_k(t)=AhF_k(t)$ we conclude that Lagrangians (19) and (13) are the same. Coincidence of the Lagrangians means that the equations of motion are the same. Hence, the variational approach helps to compose discrete analogues of the distributed systems.

5. Discussions and Conclusions

Let us consider advantages and possible limitations of the variational approach to the method of lines in comparison with the traditional one.

It is a remarkable fact that composition of two approximate numerical methods in the variational approach, namely the rectangle integration rule with accuracy $O\left[h^2\cdot\frac{\partial\Lambda(t,x)}{\partial x}\right]$ and first derivative calculation with accuracy $O\left[h^2\cdot\frac{\partial^2 u(t,x)}{\partial x^2}\right]$, gives us the same result as application of the finite difference scheme of accuracy $O\left[h^4\cdot\frac{\partial^4 u(t,x)}{\partial x^4}\right]$ to calculation of the second derivative in the standard method of lines. It means that we can deal with the derivatives of lower order in the variational approach which is the *first* obvious *advantage* of the described method.

In the example of application of the variational method we considered the simplest case of longitudinal vibrations of a bar with constant parameters considered in the frames of classical theory. Hence, numbers of terms in the Lagrangian density and in the equation were the same. If we assume now that geometrical (area of cross-section $A=A(x)$) and physical parameters (mass density $\rho=\rho(x)$ and modulus of elasticity $E=E(x)$) are variable the equation of motion is as follows (compare with (4)):

$$\rho(x)A(x)\frac{\partial^2 u(t,x)}{\partial t^2}-E(x)A(x)\frac{\partial^2 u(t,x)}{\partial x^2}-\frac{\partial\left[E(x)A(x)\right]}{\partial x}\frac{\partial u(t,x)}{\partial x}=F(t,x) \tag{20}$$

and hence, we have the four-terms equation. Moreover, we also need to calculate derivative $\frac{\partial\left[E(x)A(x)\right]}{\partial x}$ of product of the variable parameters. It can give additional numerical errors of calculation especially if these parameters are given by tables or drawings. Keep in mind that in this case the number of terms in the Lagrangian density is still equals to three and no differentiations of the geometrical and physical parameters are needed. This is a serious *second advantage* of the variational approach to the method of lines.

Let us consider how the abovementioned first and second advantages are realized in the models which need calculation of derivatives of order higher than two. As examples we consider the Rayleigh-Love and Rayleigh-Bishop models of longitudinal vibration of bars with variable geometrical (area of cross-section $A=A(x)$, polar moment of inertia $I_p=I_p(x)$) and physical (mass density $\rho=\rho(x)$, modulus of elasticity $E=E(x)$, shear modulus $G=G(x)$ and Poisson's ratio $\eta=\eta(x)$ parameters. For the sake of simplicity we consider the situation of free vibration of the bars.

In the *Rayleigh-Love model* the Lagrangian is as follows:

$$L_{R-L}(t)=\int_0^l \frac{1}{2}\left\{\begin{matrix}\rho(x)A(x)\left[\frac{\partial u(t,x)}{\partial t}\right]^2-E(x)A(x)\left[\frac{\partial u(t,x)}{\partial x}\right]^2\\ +\rho(x)\eta^2(x)I_p(x)\left[\frac{\partial^2 u(t,x)}{\partial t\partial x}\right]^2\end{matrix}\right\}dx \tag{21}$$

Hence, there are three terms in the Lagrangian density and the variational approach to the method of lines needs a finite difference representation of the first partial derivative $\frac{\partial u(t,x)}{\partial x}$.

Corresponding equation is:

$$\begin{gathered}\rho(x)A(x)\frac{\partial^2 u(t,x)}{\partial t^2}-E(x)A(x)\frac{\partial^2 u(t,x)}{\partial x^2}-\frac{\partial\left[E(x)A(x)\right]}{\partial x}\frac{\partial u(t,x)}{\partial x}\\ -\rho(x)\eta^2(x)I_p(x)\frac{\partial^4 u(t,x)}{\partial t^2\partial x^2}-\frac{\partial\left[\rho(x)\eta^2(x)I_p(x)\right]}{\partial x}\frac{\partial^3 u(t,x)}{\partial t^2\partial x}=0\end{gathered} \tag{22}$$

Hence, there are five terms in this equation. In the traditional approach to the method of lines formulation it is necessary to use finite difference representation of both first and second derivatives ($\frac{\partial u(t,x)}{\partial x}$ and $\frac{\partial^2 u(t,x)}{\partial x^2}$). Moreover it is necessary to calculate derivatives of products of the geometrical and physical parameters $\frac{\partial\left[E(x)A(x)\right]}{\partial x}$ and $\frac{\partial\left[\rho(x)\eta^2(x)I_p(x)\right]}{\partial x}$ (there are no differentiations of the parameters in the Lagrangian density (21)).

In the *Rayleigh-Bishop model* the Lagrangian is as follows:

$$L_{R-B}(t)=\int_0^l \frac{1}{2}\left\{\begin{array}{l}\rho(x)A(x)\left[\frac{\partial u(t,x)}{\partial t}\right]^2-E(x)A(x)\left[\frac{\partial u(t,x)}{\partial x}\right]^2\\+\rho(x)\eta^2(x)I_p(x)\left[\frac{\partial^2 u(t,x)}{\partial t\partial x}\right]^2-G(x)\eta^2(x)I_p(x)\left[\frac{\partial^2 u(t,x)}{\partial x^2}\right]^2\end{array}\right\}dx \quad (23)$$

Hence, there are four terms in the Lagrangian density and the variational approach to the method of lines needs a finite difference representation of the first and second partial derivatives $\frac{\partial u(t,x)}{\partial x}$ and $\frac{\partial^2 u(t,x)}{\partial x^2}$.

Corresponding equation is:

$$\begin{aligned}&\rho(x)A(x)\frac{\partial^2 u(t,x)}{\partial t^2}-E(x)A(x)\frac{\partial^2 u(t,x)}{\partial x^2}-\frac{\partial\left[E(x)A(x)\right]}{\partial x}\frac{\partial u(t,x)}{\partial x}\\&-\rho(x)\eta^2(x)I_p(x)\frac{\partial^4 u(t,x)}{\partial t^2\partial x^2}-\frac{\partial\left[\rho(x)\eta^2(x)I_p(x)\right]}{\partial x}\frac{\partial^3 u(t,x)}{\partial t^2\partial x}\\&+\rho G(x)\eta^2(x)I_p(x)\frac{\partial^4 u(t,x)}{\partial x^4}+2\frac{\partial\left[G(x)\eta^2(x)I_p(x)\right]}{\partial x}\frac{\partial^3 u(t,x)}{\partial x^3}\\&+\frac{\partial^2\left[G(x)\eta^2(x)I_p(x)\right]}{\partial x^2}\frac{\partial^2 u(t,x)}{\partial x^2}=0\end{aligned} \quad (24)$$

Hence, there are eight terms in this equation. In the traditional approach to the method of lines formulation it is necessary to use finite difference representation of the first, second, third and fourth derivatives ($\frac{\partial u(t,x)}{\partial x}$, $\frac{\partial^2 u(t,x)}{\partial x^2}$, $\frac{\partial^3 u(t,x)}{\partial x^3}$ and $\frac{\partial^4 u(t,x)}{\partial x^4}$). Moreover it is necessary to calculate derivatives of products of the geometrical and physical parameters $\frac{\partial\left[E(x)A(x)\right]}{\partial x}$, $\frac{\partial\left[\rho(x)\eta^2(x)I_p(x)\right]}{\partial x}$, $\frac{\partial\left[G(x)\eta^2(x)I_p(x)\right]}{\partial x}$ and $\frac{\partial^2\left[G(x)\eta^2(x)I_p(x)\right]}{\partial x^2}$.

The *third advantage* of the variational approach to the method of lines is obvious in the case of the vibration analysis of a stepped structure. In this case it is necessary to decompose the structure into several relatively smooth sections, consider the system of partial differential equations corresponding to each section, make finite difference discretization of each equation and formulate continuity conditions (both displacements and, if necessary, combinations of displacements and strains). This is a very tedious and awkward procedure. Vice versa, in the variational approach it is possible to exploit the additivity of the Lagrangian and consider only the continuity of displacements at junctions of the neighbour sections).

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