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## Progress in the analysis of non-axisymmetric wave propagation in a homogeneous solid circular cylinder of a piezoelectric transversely isotropic material

Arthur G. Every<sup>a,\*</sup>, Michael Y Shatalov<sup>b,c</sup>, Alfred S. Yenwong-Fai<sup>a,b</sup>

<sup>a</sup> School of Physics, University of the Witwatersrand, PO WITS 2050, South Africa

<sup>b</sup> SST of Material and Manufacturing, CSIR, P O Box 395, Pretoria 0001, South Africa

<sup>c</sup> Department of Mathematics and Statistics Of Tshwane University of Technology, P.B.X680, Pretoria, 0001, FIN-40014, South Africa

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### Abstract

Non-axisymmetric waves in a free homogeneous piezoelectric cylinder of transversely isotropic material with axial polarization are investigated on the basis of the linear theory of elasticity and linear electromechanical coupling. The solution of the three dimensional equations of motion and quasi-electrostatic equation is given in terms of seven mechanical and three electric potentials. The characteristic equations are obtained through the application of the mechanical and two types of electric boundary conditions at the surface of the cylinder. A convenient method of calculating dispersion curves and phase velocities is discussed, and resulting curves are presented for propagating and evanescent waves for the piezoelectric ceramic material PZT-4 for non-axisymmetric modes of circumferential wave number  $m = 1$ . It is observed that the dispersion curves are sensitive to the type of the imposed boundary conditions as well as to the strength of the electromechanical coupling.

*Keywords:* Non-axisymmetric wave propagation; piezoelectric; transverse isotropy; dispersion curves

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### 1. Introduction

This paper is concerned with the analysis of harmonic wave propagation in an homogeneous circular cylinder of a transversely isotropic piezoelectric solid. The focus is on non-axisymmetric modes with non-zero circumferential wave number  $m=1$ . Such modes are of interest for a variety of reasons. It is well known that in the use of guided waves to non-destructively characterize defects in cylinders, the reflected modes from such defects are generally non-axisymmetric in nature. It is envisaged that results obtained in this paper can also serve as a basis in the design of piezoelectric transducers, for which standing wave modes resulting from reflections of travelling waves at cross-section boundaries of the cylinder play an important role. Further, the broad development of the finite element

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\* Corresponding author. Tel.: +27-11-717-6823; fax: +27-11-717-6879.  
E-mail address: [arthur.every@wits.ac.za](mailto:arthur.every@wits.ac.za).

method (FEM) and boundary element method (BEM) is very much dependent for validation on reference results in the form of exact model solutions, such as those provided here.

The study of wave propagation in systems with cylindrical geometry traces back to the early work of Pochhammer [1] and Chree [2], and is treated in standard texts such as Achenbach [3], Graff [4], and Rose [5]. Mirsky [6] investigated the problem of non-axisymmetric wave propagation in transversely isotropic circular solid and hollow cylinders. Other contributors to the subject include Nayfeh and Nagy [7], Berliner and Solecki [8], Niklasson and Datta [9] and Honarvar, *et. al.* [10]. Several papers have been devoted to numerical and finite element investigations of piezoelectric cylinders, including Shatalov and Loveday [11] and Bai *et. al.* [12].

Our approach parallels that of Mirsky [6] and Berliner and Solecki [8], and is also similar to that of Winkel *et. al.* [13]. In our approach the three dimensional equations of elastodynamics together with Gauss' law for the electric field are solved, and by imposing the mechanical and electric boundary conditions on the cylindrical surface, the dispersion relation is obtained from the vanishing of a determinant of the fourth order. Our results coincide with the classical results of Mirsky [6] and Berliner and Solecki [8] for a non-piezoelectric transversely isotropic cylinder.

## 2. Analytic formulation and solution

In this section we set out the equations of motion and boundary conditions, and obtain solution of the problem of wave propagation in a transversely isotropic piezoelectric cylinder within the framework of the following assumptions and approximations: linear elasticity, linear constitutional model of piezoelectricity, quasi-static approximation of the electric field, axial polarization of the piezoelectric material, absence of free charges, boundary and body forces. The axis  $Oz$  coincides with the axis of the cylinder,  $r, \theta, z$  are respectively the radius, polar angle and axial coordinate, and  $u, v, w$  are the corresponding radial, tangential and axial displacements.

Navier's equations of motion and Gauss' law in cylindrical coordinates are:

$$\begin{aligned} \frac{\partial \sigma_1}{\partial r} + \frac{1}{r} \frac{\partial \sigma_6}{\partial \theta} + \frac{\partial \sigma_5}{\partial z} + \frac{\sigma_1 - \sigma_2}{r} &= \rho \ddot{u}, \\ \frac{\partial \sigma_6}{\partial r} + \frac{1}{r} \frac{\partial \sigma_2}{\partial \theta} + \frac{\partial \sigma_4}{\partial z} + \frac{2\sigma_6}{r} &= \rho \ddot{v}, \\ \frac{\partial \sigma_5}{\partial r} + \frac{1}{r} \frac{\partial \sigma_4}{\partial \theta} + \frac{\partial \sigma_3}{\partial z} + \frac{\sigma_5}{r} &= \rho \ddot{w}, \\ \frac{\partial D_1}{\partial r} + \frac{D_1}{r} + \frac{1}{r} \frac{\partial D_2}{\partial \theta} + \frac{\partial D_3}{\partial z} &= 0, \end{aligned} \quad (1)$$

where  $\boldsymbol{\sigma}$  is the stress and  $\mathbf{D}$  the electric displacement. The coupled constitutive equations of the system are:

$$\begin{aligned} \sigma_1 &= c_{11}^E S_1 + c_{12}^E S_2 + c_{13}^E S_3 - e_{31} E_3, \\ \sigma_2 &= c_{12}^E S_1 + c_{11}^E S_2 + c_{13}^E S_3 - e_{31} E_3, \\ \sigma_3 &= c_{13}^E (S_1 + S_2) + c_{33}^E S_3 - e_{33} E_3, \\ \sigma_4 &= c_{44}^E S_4 - e_{15} E_2, \quad \sigma_5 = c_{44}^E S_5 - e_{15} E_1, \\ \sigma_6 &= c_{66}^E S_6, \quad D_1 = \varepsilon_{11}^S E_1 + e_{15} S_5, \quad D_2 = \varepsilon_{11}^S E_2 + e_{15} S_4, \\ D_3 &= \varepsilon_{33}^S E_3 + e_{31} (S_1 + S_2) + e_{33} S_3, \end{aligned} \quad (2)$$

where  $\mathbf{S}$  is the strain,  $\mathbf{E}$  the electric field,  $c_{11}^E, \dots, c_{66}^E$  the elastic stiffnesses at constant electric field,  $e_{15}, e_{31}, e_{33}$  the piezoelectric constants and  $\epsilon_{11}^S, \epsilon_{33}^S$  are the dielectric constants. The electric field is given by:

$$E_1 = -\frac{\partial \phi}{\partial r}, \quad E_2 = -\frac{1}{r} \frac{\partial \phi}{\partial \theta}, \quad E_3 = -\frac{\partial \phi}{\partial z}, \tag{3}$$

where  $\phi$  is the electric potential.

We seek solutions to equations (1) to (3) in the form of harmonic waves travelling along the  $z$ -axis, and expressed in terms of displacement and electric potentials. For an isotropic non-piezoelectric solid it is customary to invoke Helmholtz's theorem and express the 3 component displacement field as the gradient of a scalar potential  $\phi$  plus the curl of a vector potential  $\Psi$ , and reduce the number of independent potential functions from 4 to 3 with a constraint such as  $\nabla \cdot \Psi = 0$ . For a transversely isotropic solid, since the  $X$  and  $Y$  axes are equivalent to each other but not to the  $z$  axis, a separate scalar potential  $\chi$  is required for the  $z$  displacement, and to keep the number of potential functions at 3 it is expedient to set the  $X$  and  $Y$  components of  $\Psi$  to zero, retaining just a single component  $\psi$ . In the case of a piezoelectric medium, there is also the electric potential  $\phi$  to consider. Inserting these quantities in equations (1) to (3), one arrives at a set of PDE's which decouple if one sets  $\chi = \eta \phi$  and  $\psi = \mu \phi$ , the coefficients of proportionality  $\eta$  and  $\mu$  having three sets of values that emerge from the solutions of a cubic equation, and each associated with a different set of scalar potentials  $\phi, \chi$  and  $\psi$ . There are thus 10 potential functions altogether, but with only four being independent. Armed with this knowledge in hindsight, we therefore embark at the outset with the following representation of  $\mathbf{u}, \mathbf{v}, \mathbf{w}, \phi$  in terms of 10 potentials  $\Phi_j = \Phi_j(r, \theta)$ :

$$u = \left[ \sum_{j=1}^3 \frac{\partial \Phi_j}{\partial r} + \frac{1}{r} \frac{\partial \Phi_4}{\partial \theta} \right] e^{i(\omega t + kz)}, \quad v = \left[ \frac{1}{r} \sum_{j=1}^3 \frac{\partial \Phi_j}{\partial \theta} - \frac{\partial \Phi_4}{\partial r} \right] e^{i(\omega t + kz)},$$

$$w = \left[ \sum_{j=5}^7 \Phi_j \right] e^{i(\omega t + kz)}, \quad \phi = \left[ \sum_{j=8}^{10} \Phi_j \right] e^{i(\omega t + kz)}, \tag{4}$$

where  $\omega$  is the angular frequency, and  $k$  is the wave number (real for propagating and imaginary or complex for evanescent waves).

After substituting Eqn. (4) in (3) and further in (2) and (1), the following system of equations is obtained:

$$\frac{\partial}{\partial r} \left\{ \sum_{j=1}^3 [c_{11}^E \nabla^2 \Phi_j + (\rho \omega^2 - k^2 c_{44}^E) \Phi_j] + ik(c_{15}^E + c_{31}^E) \sum_{j=8}^{10} \Phi_j + ik(e_{15} + e_{31}) \sum_{j=8}^{10} \Phi_j \right\}$$

$$+ \frac{1}{r} \frac{\partial}{\partial \theta} \left\{ c_{66}^E \nabla^2 \Phi_4 + (\rho \omega^2 - k^2 c_{44}^E) \Phi_4 \right\} = 0,$$

$$\frac{1}{r} \frac{\partial}{\partial \theta} \left\{ \sum_{j=1}^3 [c_{11}^E \nabla^2 \Phi_j + (\rho \omega^2 - k^2 c_{44}^E) \Phi_j] + ik(c_{13}^E + c_{44}^E) \sum_{j=8}^{10} \Phi_j + ik(e_{15} + e_{31}) \sum_{j=8}^{10} \Phi_j \right\}$$

$$- \frac{\partial}{\partial r} \left\{ c_{66}^E \nabla^2 \Phi_4 + (\rho \omega^2 - k^2 c_{44}^E) \Phi_4 \right\} = 0, \tag{5}$$

$$ik(c_{15}^E + c_{31}^E) \sum_{j=1}^3 \nabla^2 \Phi_j + \sum_{j=5}^7 [c_{44}^E \nabla^2 \Phi_j + (\rho \omega^2 - k^2 c_{33}^E) \Phi_j] + \sum_{j=8}^{10} [e_{15} \nabla^2 \Phi_j - k^2 e_{33} \Phi_j] = 0$$

$$ik(e_{15} + e_{31}) \sum_{j=1}^3 \nabla^2 \Phi_j + \sum_{j=5}^7 [e_{15} \nabla^2 \Phi_j - k^2 e_{33} \Phi_j] - \sum_{j=8}^{10} [e_{11} \nabla^2 \Phi_j - k^2 e_{33} \Phi_j] = 0$$

where  $\nabla^2$  is the 2D Laplacian in polar coordinates.

The first two equations of the system are satisfied if the quantities in the braces are zero, i.e.

$$\sum_{j=1}^3 [c_{11}^E \nabla^2 \Phi_j + (\rho \omega^2 - k^2 c_{44}^E) \Phi_j] + ik(c_{15}^E + c_{31}^E) \sum_{j=8}^{10} \Phi_j + ik(e_{15} + e_{31}) \sum_{j=8}^{10} \Phi_j = 0, \quad c_{66}^E \nabla^2 \Phi_4 + (\rho \omega^2 - k^2 c_{44}^E) \Phi_4 = 0 \tag{6}$$

The second equation of (6) is a Helmholtz equation for  $\Phi_4$ ,

$$\nabla^2 \Phi_4 + \xi_4^2 \Phi_4 = 0, \quad \xi_4^2 = (\rho \omega^2 - k^2 c_{44}^E) / c_{66}^E, \tag{7}$$

decoupled from the other potentials.

Compatibility between the first equation of (6) and the third and fourth equations of (5) can be achieved by setting

$$\begin{aligned} \Phi_j &= \eta_{j-4} \Phi_{j-4} \quad (j = 5, 6, 7), \\ \Phi_l &= \mu_{l-7} \Phi_{l-7} \quad (l = 8, 9, 10), \end{aligned} \tag{8}$$

from which it follows that that the potentials  $\Phi_j$  satisfy Helmholtz equations

$$\nabla^2 \Phi_j + \xi_j^2 \Phi_j = 0 \quad (j = 1, 2, 3), \tag{9}$$

with the  $\xi_j$  being determined by the bi-cubic equation:

$$(\xi_j^2)^3 + b_1 (\xi_j^2)^2 + b_2 (\xi_j^2) + b_3 = 0, \tag{10}$$

where  $b_{1,2,3}$  are simple functions of the materials constants,  $\omega$  and  $k$ .

In the general case there are three roots to Eqn. (10) and

$$\begin{aligned} \eta_j &= -\frac{i \xi_j^4 c_{11}^E e_{15} + \xi_j^2 [k^2 B_6 - (\rho \omega^2) e_{15}] + k^2 (k^2 c_{44}^E - \rho \omega^2) e_{33}}{k^2 B_7 + \xi_j^2 B_8 + (\rho \omega^2) B_9}, \\ \mu_j &= \frac{\xi_j^2 (\eta_j e_{15} - ik B_9) + k^2 \eta_j e_{33}}{\xi_j^2 e_{11} + k^2 e_{33}}, \quad (j = 1, 2, 3), \end{aligned} \tag{11}$$

where  $B_{6,7,8}$  are simple functions of the materials constants.

Solutions of the Helmholtz equations (7) and (9) take the form:

$$\begin{aligned} \Phi_j(r, \theta) &= A_j W_m(\xi_j r) \cos(m\theta), \quad (j = 1, 2, 3), \\ \Phi_4(r, \theta) &= A_4 W_m(\xi_4 r) \sin(m\theta), \end{aligned} \tag{12}$$

where for the solid cylinder we are considering,  $W_m(\xi r) = J_m(\xi r)$  is a Bessel function of the first kind if  $\xi$  is real or complex, and  $W_m(\xi r) = I_m(|\xi| r)$  is a modified Bessel function of the first kind if  $\xi$  is pure imaginary,  $m$  is the integer circumferential wave number, and the  $A_j$  (real or complex) are amplitudes to be determined from the boundary conditions. Solutions in the form of Bessel and modified Bessel functions of the second kind are disregarded for a solid cylinder because of their singular behavior at the origin.

The mechanical boundary conditions we consider are the absence of external forces on the cylindrical boundary at radius  $r = a$ :

$$\sigma_1|_{r=a} = \sigma_5|_{r=a} = \sigma_6|_{r=a} = 0, \tag{13}$$

and there are two types of electric boundary conditions:

$$D|_{r=a} = 0 \quad \text{or} \quad \phi|_{r=a} = 0, \tag{14}$$

where the first expression corresponds to the open-circuit condition and the second expression to the close-circuit condition.

After substitution (12) into (4), (3), and (2) and then (13) and either of the two conditions (14), we obtain a system of four linear homogeneous algebraic equations for the unknown amplitudes  $A_{1, \dots, 4}$ . The coefficients  $A_{ij}$  of

these equations are functions of the materials constants,  $\omega$ ,  $k$ ,  $m$  and  $a$ . These equations have a non-trivial solution if and only if their determinant vanishes:

$$Det = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0. \tag{15}$$

In the limit of zero electro-mechanical coupling  $e_{15} = e_{31} = e_{33} = 0$ , this result reduces to that of Berliner and Solecki [8] for a non piezoelectric transversely isotropic cylinder.

### 3. Numerical results and discussion

In this section we present dispersion curves for non-axisymmetric waves with circumferential wave number  $m = 1$  obtained from equation (15) for the piezoelectric ceramic, PZT-4. Rather than employing a traditional root finding algorithm, we display the logarithm of the modulus of the determinant (15) on a 2D mesh of points  $(k, \omega)$ . At those points where the real and imaginary parts of determinant (15) are close to zero, negative spikes occur which provide a good approximate representation of the dispersion curves. A problem encountered is that the roots of the characteristic arguments  $\xi_j = 0, j = 1, \dots, 4$  also show up in the plots as obvious artifacts, but these are readily removed.

Dispersion curves of bending waves ( $m = 1$ ) in the cylinder with the short-circuit lateral (cylindrical) surface are depicted in Fig. 1 for dimensionless frequency  $\Omega = a\omega/V_s$  in the range  $[0, 14]$ , where  $V_s = \sqrt{c_{44}^E/\rho}$ , and  $a$  is the outer radius of the cylinder, and for real values (propagating waves) and pure imaginary values (evanescent waves) of the  $k \cdot a$  in the range  $\text{Re}(k \cdot a), \text{Im}(k \cdot a) \in [0, 8]$ .

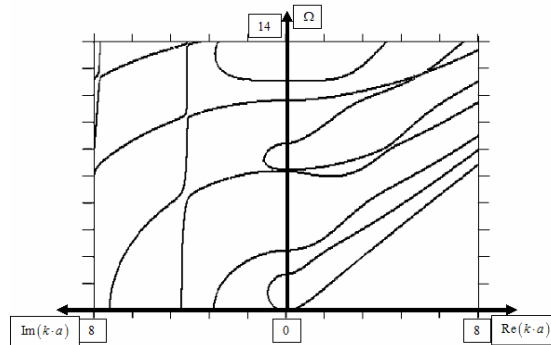


Fig.1 ( $m = 1$ ) Dispersion relation for PZT-4 cylinder with short-circuit lateral surface conditions.

The lowest dispersion curve for propagating waves (real  $k$ ) tends asymptotically to the surface wave mode. It is joined through the domain of evanescent waves (imaginary  $k$ ) to the second propagating branch, which tends asymptotically to the shear wave mode. These results are consistent with the finite element calculations of Bai *et al.* [12] for wave propagation in a hollow piezoelectric cylinder. Dispersion curves of bending waves ( $m = 1$ ) under

open-circuit conditions for the cylindrical surface are shown in Fig. 2. It is obvious from comparison of the two figures, that the electric boundary conditions substantially influence both propagating and evanescent waves.

In Fig. 3 a conceptual case of greatly reduced electro-mechanical coupling coefficients is shown.

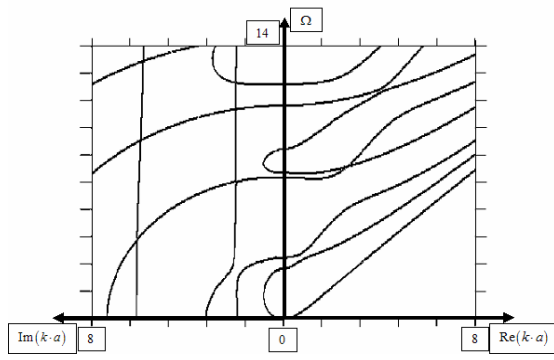


Fig.2 ( $m = 1$ ) Dispersion relation for PZT-4 cylinder with open-circuit lateral surface conditions.

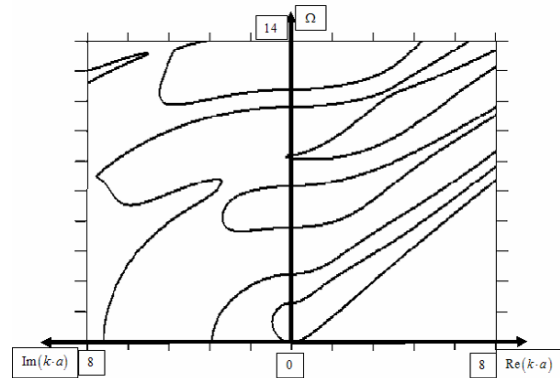


Fig.3 ( $m=1$ ) Dispersion relation for PZT-4 cylinder with reduced electro-mechanical coupling  
 $(m=1, e_{15} = e_{31} = 0, e_{33} = 0.001e_{33(PZT-4)})$

It is observed from Figs. 1, 2 and 3 that the lowest fundamental branch for bending ( $m = 1$ ) waves is relatively insensitive to the nature of the electric boundary conditions, and also not very sensitive to the strength of electro-mechanical coupling of the material. The higher order modes are more sensitive to the nature of the electric boundary condition and also to the strength of the electro-mechanical cross-coupling. The PZT-4 cylinder under short-circuit conditions exhibits a negative slope in the fourth branch in quite a broad range of wave numbers (Fig. 1). Substantial dependence on electric boundary conditions is obvious in the behavior of the dispersion curves for the evanescent waves.

A more extensive treatment of this problem is to be published elsewhere. [14]

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