Reducing Condition Number by Appropriate Current Decomposition on a Multiplet of Several Wires

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***Abstract*—This paper discusses a numerical investigation in connection with the dependency of the condition number of the impedance matrix on the decomposition of current on a junction with several attached wires (multiplet). It is shown that the condition number is frequency- and wire selection dependant, and it is possible to minimize it, and that this helps to minimize the condition number of the system this multiplet belongs to. It is also believed that the same approach is applicable not just for wire modelling but also for surface elements.**

Keywords-moment method; condition number; wire modeling

I. Introduction

The *moment method* (MM) [1], [2], [3] is a popular method of solving electromagnetic problems and transforms integral form of Maxwell equations into a system of linear algebraic equations. This is accomplished by approximating the unknown current distribution as a superposition of known *basis functions* with unknown constant coefficients, substituting them into the integral equations, and weighting the result with *testing functions*. In this paper, the use of Galerkin's scheme is assumed, where the testing functions are the same as basis functions.

Each basis function is assigned to a specific part of the geometrical (mechanical) structure. This paper assumes that the basis functions are doublets [3], i.e. are made up of a pair of *piecewise linear* (PWL) functions defined in [2] as roof-tops, and that geometry may be approximated by a set of thin wires. It is assumed that the thin wire approximation conditions [3] are satisfied. The doublets may partially overlap.

The solution of the resultant system of linear equations is dependant on the condition of its matrix referred in MM to as the *impedance matrix*. The elements of this matrix represent the strength of coupling between the pairs basis functions. The diagonal elements of the matrix represent self-coupling terms, discussed in more detail in [2]. A well conditioned matrix will permit accurate results, as may be noted from the following expression [4], [5] interconnecting the relative accuracy of solution $\|\mathbf{y}-\mathbf{x}\|/\|\mathbf{x}\|$ of the perturbed system of linear equations to the accuracy in specifying the system of linear equations:

$$\frac{\|\mathbf{y}-\mathbf{x}\|}{\|\mathbf{x}\|} \le \frac{1}{1-r}\kappa(\mathbf{Z})\left(\frac{\|\delta\mathbf{Z}\|}{\|\mathbf{Z}\|}+\frac{\|\delta\mathbf{V}\|}{\|\mathbf{V}\|}\right), \tag{1}$$

where it is assumed that quantity $r=\|\delta\mathbf{Z}\|\cdot\|\mathbf{Z}^{-1}\|<1$, $\kappa(\mathbf{Z})\equiv\|\mathbf{Z}\|\cdot\|\mathbf{Z}^{-1}\|$ is the condition number of the non-singular matrix $\mathbf{Z}$, the notation $\|\mathbf{Z}\|$ denotes the norm of the matrix $\mathbf{Z}$, $\mathbf{x}$ is the column vector of the exact solution, $\mathbf{y}$ is the column vector of the perturbed solution, and the symbol δ denotes the perturbation terms ($\delta\mathbf{Z}$ is perturbation of the impedance matrix $\mathbf{Z}$ and $\delta\mathbf{V}$ is perturbation of the excitation column vector $\mathbf{V}$). A condition number near unity normally corresponds to a well-conditioned system, whilst a high condition number may indicate an ill-conditioned system [5].

The expression (1) shows that the error in the solution increases with an increase in the condition matrix. Even if a linear system is solved by the use of an iterative approach [6], an ill-conditioned impedance matrix will lead to a slow convergence [6]. Thus, it is important to try to minimize the condition number of the impedance matrix.

A complex system of interconnected wires may be decomposed into junctions. A junction may have an arbitrary number of wires segments attached to it. An individual junction with the attached wire segments may be considered as a three-dimensional object or a sub-system of all wires. The model of current distribution on the wire segments of the junction can be composed of a superposition of doublets and referred to as multiplet [3]. Such a subsystem will also have an impedance matrix associated with it. This impedance matrix is a sub-matrix or a part of the impedance matrix for the complete system of interconnected wires.

The overlap of the doublet basis functions defined on the wire segments that belong (are connected) to a junction leads to the possibility to use Kirchhoff's current law [3] to eliminate one "dependent" unknown from the number of unknowns describing the model of current distribution on the junction [3]. This is for instance used to in the roof-top [2] and doublet [3] basis functions to automatically satisfy the current continuity condition.

This paper shows that the condition number for the impedance matrix for the wire segments connected to a junction is frequency dependent and that the condition number may be minimized by an appropriate selection of the wire segment and respective basis function to be considered as the dependent unknown. No previous publications on this topic have been identified by the author.

The paper is arranged as follows. Section II discusses the formulation of the problem, method of study, samples of geometry used for the study, and the results obtains. Section III discusses the results and several methods of minimization of condition number, and concludes the paper.

II. Problem Formulation, Method and Results

A junction with *N* wire segments attached, will use *N* doublets to make up a multiplet, and possibly basis functions of higher order, singletons [3], to model the total current distribution. This paper focuses on the doublets, as the singletons do not contribute to the satisfaction of the Kirchhoff's current continuity law. Only *N*-1 doublets correspond to the independent unknowns/variables, and one unknown may be taken as dependent and computed via the independent variables once those are found via MM.

The choice of the dependent unknown is usually considered as arbitrary. Some programs, for example, take the first unknown in the list of unknowns associated with a junction as the dependent one.

A. Method of investigation

The method used in this work to investigate the dependence of the condition number on the choice of dependant unknown (i.e. on the choice of doublet and associated wire segment; hereinafter, this wire segment is referred to as the *reference wire*) includes the following steps:

Loop through all the wire segments attached to the junction

Select the dependent variable on the current wire segment

Compute impedance matrix and condition matrix

End of loop

After the computations are done, the condition numbers obtained are inter-compared, enabling to analyse the length of the wire segment associated. The process is repeated at several frequencies, when necessary.

In the following, several examples to illustrate the method and show the results are considered. The subsection A introduces the method of numerical analysis used to probe the dependence of the condition number against the choice of the reference wire at a very low frequency. The next subsection shows a more comprehensive example covering several frequencies.

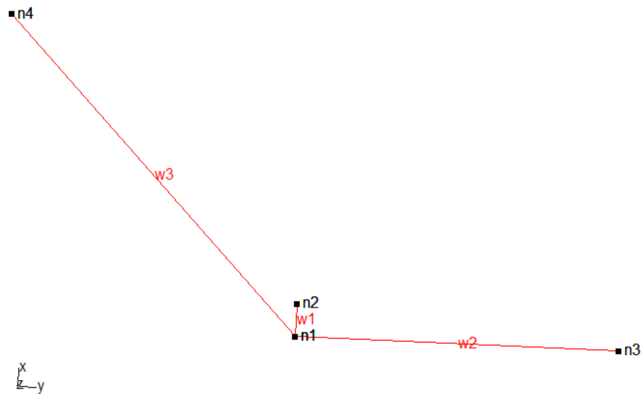

Figure 1. Sample structure used for investigation of the dependence of the condition number on the choice of the reference wire. Model with three wires: each one is at a right angle to each of the other two wires. Note on the perspective: the z- axis on this figure is close to but not perpendicular to the page (not easy to see as the angle is about 8 degrees off), and so the x- and y- axes are not exactly parallel to the page.

TABLE I. Definitions for node coordinates

Node no.	X, m	Y, m	Z, m	Comment
1	0	0	0	Common node for all wires
2	1	0	0	The shortest dist. to node 1
3	0	10	0	
4	0	0	100	The longest dist. to node 1

TABLE II. Defining the wires via nodes from Table I

Wire no.	Nodes defining the wire (beginning, end)
w1	1,2
w2	1,3
w3	1,4

TABLE III. Impedance Matrix Condition Number for Various Assignments of Wires to Nodes

Configu-ration No.	Nodes of wires w1, w2, and w3: as (beginning, end)	Condition number	Comment
A	(1,2) (1,3) (1,4)	**27.84**	Shortest wire is the reference one
B	(1,3) (1,2) (1,4)	8.136	Middle is the reference one
C	(1,2) (1,4) (1,3)	27.84	Shortest wire is the reference one
D	(1,4) (1,2) (1,3)	**7.019**	Longest wire is the reference one
E	(1,4) (1,3) (1,2)	**7.019**	Longest wire is the reference one
F	(1,3) (1,4) (1,2)	8.136	Middle is the reference one

B. Quasistatic case with only three wire segments

For the sake of simplicity this example includes only three wire segments and the study is performed at one frequency. The structure is shown in Fig. 2, which shows three wires of different length electrically connected at one point (junction). The dimensions of the wires and the indexing scheme are shown in Tables I and II, respectively. The length of wires was made different on purpose, to highlight the effects.

The method explained in the previous subsection was applied. The results are shown in Table III. The table shows that under the quasi-static scenario, and when the reference wire is the shortest wire, this leads to the maximum (worst) condition number. On the contrary, when the reference wire is the shortest wire, this leads to the lowest (best) condition number.

C. Array of Junctions

It was expected that a complex system composed of sub-systems should gain better (lower) system condition number by optimizing (minimizing) the condition number for each individual sub-system.

In this example, this assumption is tested. The example shows a more complex geometry where an array was composed of the structures discussed in the previous subsection. The geometry is shown in Fig. 2.

The single junction with three wires already considered in the previous subsection is used as a building block (sub-system) for a larger structure. In the numerical experiments, the number of subsystems was varied from 1 to 20. Fig. 2 shows the twenty subsystems case. A subsystem case is in Fig. 1.

In each scenario with a fixed number of subsystems, the impedance matrix was computed for three different basis function assignments identified in the example from the previous subsection. In one configuration, the shortest wires, i.e. wires no 1, 4, 7, …, were used to define the common domains for halves of the doublet basis functions (as the reference wires for the respective junctions). In the next tested configuration, wires with the intermediate length, i.e. wires no 2, 5, 8, …, were used as the reference/common. In the 3rd assignment, the longest wires, i.e. wires no 3, 6, 9, …, were used as the reference/common.

The results of simulations are shown in Fig. 3. All plots there have a common feature. The condition number saturates as the size of the system increases. The only exception is observed at the frequency of 2.46 MHz. Around this frequency, a resonance is observed in each individual sub-system. The trend towards saturation is still present there, but is much less pronounced. The value of the condition number near the resonance is much higher than far outside of the resonance frequency.

This observation may help to explain the saturation of the condition number with linear growth of the system: when the second subsystem is added to the one from the previous subsection, these two subsystems start to interact, affecting current distribution in each other. As more subsystems are added, the new additions are spaced further apart from the original subsystem, and thus have less interaction with the original subsystem. In addition, the subsystems standing in-between may act as a screen. Thus, continuing to add new subsystems does not increase the complexity of interactions proportionally, resulting in saturation.

However, when resonance conditions occur (e.g. around the frequency 2.46 MHz), the interactions become more intense. In the system under consideration, most of coupling between subsystems is due to the longest wires that are parallel to each

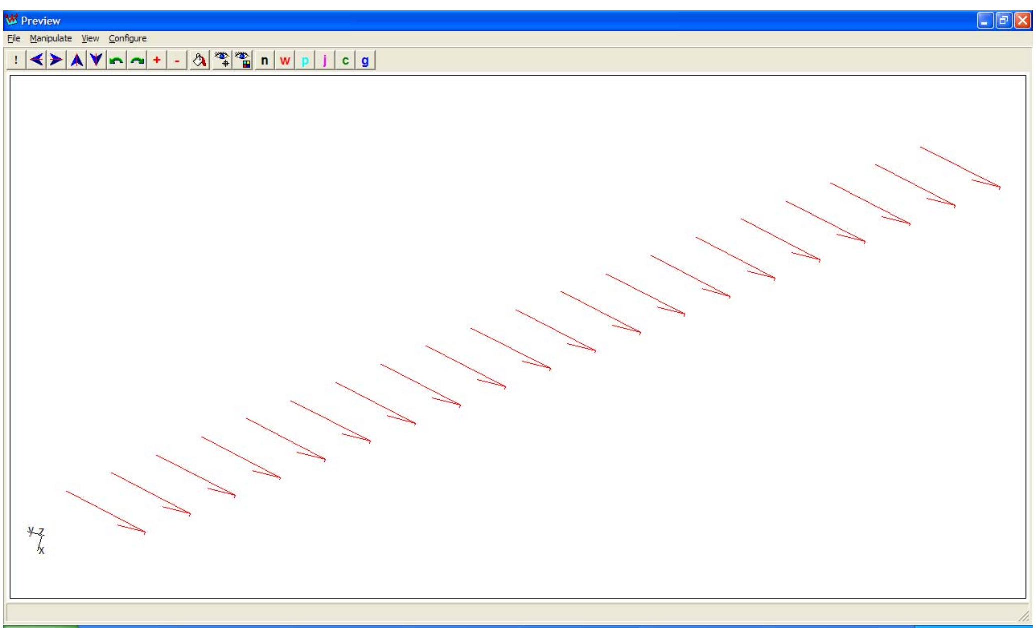

Figure 2. Making a larger system out of a single junction with three wires. The print-screen shows the basic element – a junction with 3 wires attached (lengths 1 m, 10 m, and 100 m, all perpendicular to each other) – copied 20 times to make an array. There is twenty subsystems. The spacing between the subsystems is equidistant and equals (10 m, 10 m, 10 m) for the coordinates (x,y,z), respectively.

other. The distance between the wires is approximately (102+102)1/2=14.1 m. This is close to one eighth of the wavelength λ, i.e. $\lambda/8$= 15.2 m, providing conditions to have strong interactions despite long-distance. It is thus assumed that the resultant system is a tightly coupled large system with a high condition number, as illustrated in Fig. 3b.

With regards to lowering the condition number, the results plotted in subplots of Fig. 3 confirm the assumption and conclusions made in the previous subsection: *selecting the longest wire for the reference wire minimises the condition number at low frequencies. In a similar manner, selecting the shortest wire as the reference wire minimises the condition number at higher frequencies*.

D. Additional tests

More similar tests have been carried. The length of wires was varied and the configuration of wires altered. They brought in similar conclusions, as made in the previous subsection.

III. DISCUSSIONS AND CONCLUSION

The work considered several sample scenarios of junctions with wires and an array of such junctions. The condition number was computed for various choices of the dependent variable and associated “reference” wire segment.

The results show that the choice of the reference wire in a junction with several wire segments is frequency dependent. The examples considered showed that an appropriate choice of the reference wire offered a reduction in the condition number by the factor of up to five. It is expected that even greater reduction is possible under more extreme conditions, e.g. when the difference in wire lengths is greater.

In order to minimize the condition number, the following choices are available:

a) At low frequencies, the longest wire should be selected as the reference wire.

b) At high frequencies, the shortest wire should be selected as the reference wire.

c) It is possible to find the minimum possible condition number by iterating through all the possible combinations of reference wire.

d) Alternatively, it is possible to never select the shortest or the longest wire as the reference wire. This should help to avoid the instances of excessively high condition number at any frequency, although does not necessarily provide the minimum possible value of the condition number. An additional advantage of this approach is that, unlike the option (c), it does not require any heavy computations.

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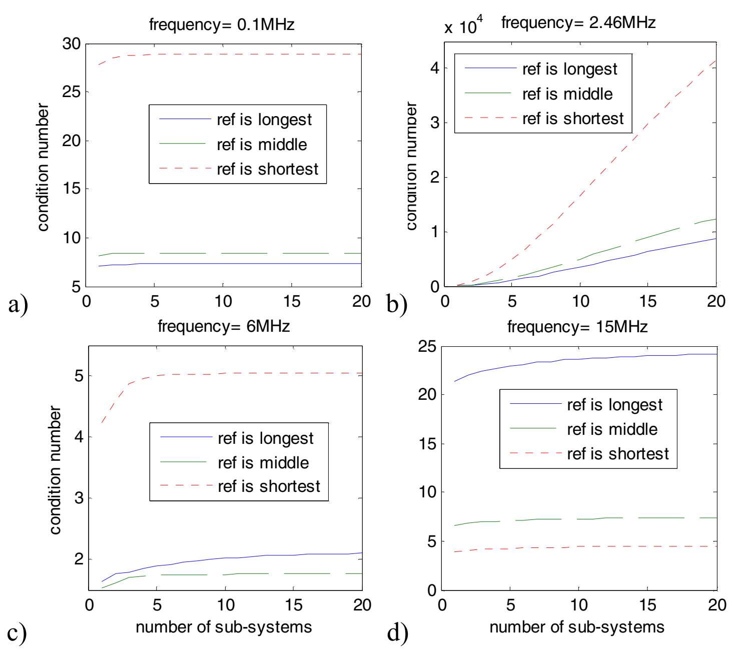

Figure 3. Condition number versus complexity of system, computed for different assignments of reference/common wires at respective junctions. The four plots display graphs for four different frequencies (from quasi-static to very high).