Parameter dependence of the decoherence of orbital angular momentum entanglement in atmospheric turbulence

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ABSTRACT

The orbital angular momentum (OAM) state of light can potentially be used to implement higher dimensional entangled systems for quantum communication. Unfortunately, optical fibers in use today support only modes with zero OAM values. Free-space quantum communication is an alternative to traditional way of communicating through optical fibers. However the refractive index fluctuation of the atmosphere gives rise to random phase aberrations on a propagating optical beam. To transmit quantum information successfully through a free-space optical channel, one needs to understand how atmospheric turbulence influences quantum entanglement. Here, we present a numerical study of the evolution of quantum entanglement between a pair of qubits. The qubits consist of photons entangled in the OAM basis. The photons propagate in a turbulent atmosphere modeled by a series of consecutive phase screens based on the Kolmogorov theory of turbulence. Maximally entangled initial states are considered, and the concurrence is used as a measure of entanglement. We show how the evolution of entanglement is influenced by various parameters such as the beam waist, the strength of the turbulence and the wavelength of the beam. We restricted our analysis to the OAM values $l = \pm 1$ and we compared our results to previous work.

1. INTRODUCTION

The orbital angular momentum (OAM) state of light has been the object of much interest within the quantum information community lately. This is mainly because it can be used to implement higher dimensional entanglement as it was noticed that one can in principle use the OAM state of photon to describe an infinite dimensional Hilbert space. A Laguerre-Gaussian mode with azimuthal index l carries an OAM of $l\hbar$ per photon.[1,2]

Quantum entanglement plays an essential role in the new technological developments, such as quantum computation, quantum cryptography, quantum metrology and quantum communication. One of the main limiting factors of these technologies is decoherence. Decoherence consists of a family of effects that occur due to the interaction between quantum systems and their environments. During that interaction, entanglement between the subsystems of the quantum system is destroyed. The use of OAM states of photons for quantum information processing presents many challenges, one of the biggest challenges is the distortion of the modes during transmission of OAM-encoded photons over large distances. Most optical fibers in use today cannot be use to transmit OAM entangled photons because these fibers only support modes with zero OAM. One can alternatively use free-space communication. However, it is crucial to understand how OAM entanglement decoheres due to atmospheric turbulence.

In this work, we present a numerical study of the effect of the different parameters (beam radius, wavelength and refractive index structure parameter) on the evolution of quantum entanglement between a pair of OAM-entangled photons propagating in a turbulent atmosphere. Some works have been done on the effect of the atmospheric turbulence on the OAM state of photons.[3–5] However, these works are based on the assumption that the overall effect of the turbulence over the propagation path can be modeled by a pure phase-only perturbation on the beam.[6] Here, we will investigate under what circumstances is that assumption valid. We model the

atmosphere with a series of consecutive phase screens separated by a distance Δz. We use Kolmogorov theory of turbulence,[7,8] and we restrict our analysis to two OAM states. We consider only qubits and the Wooter's concurrence[9] is used as a measure of quantum entanglement.

This paper is organized as follows. In Section 2, we describe the theoretical background used in this work. That includes the method we used to generate the phase screens simulating the turbulent atmosphere, the propagation process, and how we obtained the density matrix. In Section 3, we present the numerical procedure and the results. Then few conclusions are drawn in Section 4.

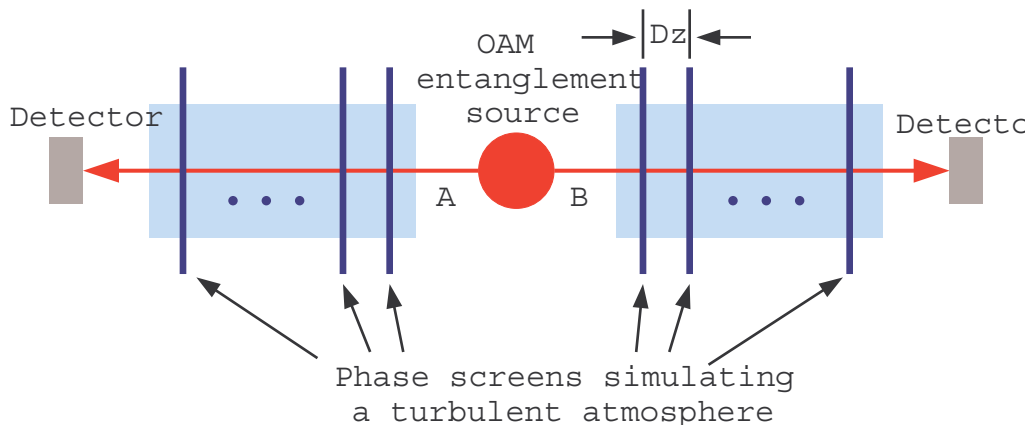## 2. THEORETICAL BACKGROUND

2.1 The split-step method

Figure 1. The source generates two photons that are entangled in OAM. Each photon is then sent through a turbulent atmosphere (modeled by a series of phase screens) toward a detector.

Our numerical simulations are based on a technique known as the split-step method.[10,11] In this method, the atmosphere is modeled by a series of phase screens separated by a distance Δz as illustrated in Fig. 1. Each phase screen represents a turbulent atmosphere layer of thickness Δz. As the beam goes through a phase screen, its phase will be distorted. After the phase screen, the beam is propagated through free space over the distance Δz between consecutive phase screens. During that propagation, the phase distortion will induce an amplitude distortion on the beam. The phase fluctuation $\theta(x, y)$ on the phase screen is related to the refractive index fluctuation $\tilde{n}(x, y, z)$ of the medium through

$$\theta(x, y) = k_0 \int_0^{\Delta z} \tilde{n}(x, y, z) \mathrm{dz}, \tag{1}$$

where k_0 is the wave number.

The common way of calculating the phase fluctuation is to use the relationship between the phase spectrum and the refractive index spectrum as explained by the following few equations. From Eq. (1), the phase correlation function can be written as[10]

$$\begin{aligned} B(x_1 - x_2, y_1 - y_2) &= k_0^2 \iint_0^{\Delta z} \langle \delta n(x_1, y_1, z_1) \delta n(x_2, y_2, z_2) \rangle \mathrm{dz}_1 \mathrm{dz}_2 \\ &= k_0^2 \Delta z A(x_1 - x_2, y_1 - y_2), \end{aligned} \tag{2}$$

where $A(x_1 - x_2, y_1 - y_2)$ is the autocorrelation function of the refractive index in the x-y plane (plane perpendicular to the propagation direction). In Eq. (2), it is assumed that Δz is larger than the correlation length

of the irregularities. This assumption allows one to write the three-dimensional autocorrelation function of the refractive index $A(x, y, z)$ as the two-dimensional function $A(x, y)$ in the x-y plane multiplied by a Dirac delta function in the z-direction. From the Wiener Khinchin theorem, the power spectral density of a random process is the Fourier transform of the corresponding autocorrelation function. We can then write

$$A(x, y) = 2\pi \int \int_{-\infty}^{\infty} \Phi_n(k_x, k_y, k_z = 0) \exp[-\mathrm{i}(k_x x + k_y y)] \mathrm{d}k_x \mathrm{d}k_y. \tag{3}$$

By substituting Eq. (3) into Eq. (2), and making use of the Wiener Khinchin theorem once more, we get

$$\Phi_\theta(k_x, k_y) = 2\pi k_0^2 \Delta z \Phi_n(k_x, k_y, k_z = 0). \tag{4}$$

Many formulations have been developed to model the spectrum of the refractive index fluctuations, including the Kolmogorov spectrum,[7,8] the Tatarskii spectrum,[8,12] the von Karman spectrum[8,10] and the modified von Karman spectrum.[8,13] The present work will focus on the Kolmogorov spectrum, which is given by

$$\Phi_n^K(k) = 0.033 C_n^2 k^{-11/3}. \tag{5}$$

The random phase field is generated by taking the inverse Fourier transform of the square root of Φ_θ multiplied by a random complex function with a Gaussian distribution in the frequency domain

$$
\begin{aligned}
\theta_1(x, y) + \mathrm{i}\theta_2(x, y) \quad &= \quad \mathcal{F}^{-1}\left\{ \frac{\chi(k_x, k_y)\left[\Phi_\theta(k_x, k_y)\right]^{1/2}}{\Delta_k} \right\} \\
&= \quad \frac{1}{\Delta_k}(2\pi k_0^2 \Delta z)^{1/2} \mathcal{F}^{-1}\left\{ \chi(k_x, k_y)\left[\Phi_n(k_x, k_y, k_z = 0)\right]^{1/2} \right\},
\end{aligned}
\tag{6}
$$

where $\mathcal{F}^{-1}$ is the two-dimensional inverse Fourier transform, $\chi(k_\perp)$ is a zero-mean Gaussian random complex function generated in the frequency domain and Δ_k is the spacing between samples in the frequency domain. This method has the advantage of generating two phases screens, each with Gaussian statistics. The transmission function of those phase screens are $T_1 = \exp(i\theta_1)$ and $T_2 = \exp(i\theta_2)$.

The whole propagation process is done as follows. Suppose we want to propagate an LG_{lp} beam through a phase screen from $z = 0$ to $z = \Delta z$ with the phase screen placed at z_0. Let $f(x, y, z = 0)$ and $f(x', y', z = \Delta z)$ be the functions describing the beam before and after propagation respectively. First we send the beam through the phase screen. This is equivalent to multiplying the input function with the transmission function T of the phase screen

$$f'(x, y) = f(x, y, z = 0)T. \tag{7}$$

After the phase screen the result is propagated through the distance Δz as follows. First we computed the angular spectrum $F(k_x, k_y)$ of $f'(x, y)$ by computing its Fourier transform, the output function is then obtained by computing the inverse Fourier transform of $F(k_x, k_y)$, multiplied with the propagation phase factor:

$$f(x', y', z = \Delta z) = \mathcal{F}^{-1}\{F(k_x, k_y)\exp(-2\pi \mathrm{i}c\Delta z)\}, \tag{8}$$

where $c = \left(1/\lambda^2 - (k_x/2\pi)^2 - (k_y/2\pi)^2\right)^{1/2}$.

2.2 Decoherence process

As illustrated in Fig. 1, the system studied consists of two photons entangled in their OAM mode. Each photon is represented by an LG mode propagating in a turbulent atmosphere. The LG mode is given in normalized cylindrical coordinates by

$$\begin{aligned}
\mathrm{LG}_{lp}(r,\phi,z) \;&=\; \langle \mathbf{r}|l,p\rangle \\
&=\; \mathcal{N}\frac{r^{|l|}\exp(il\phi)(1+\mathrm{i}t)^p}{(1-\mathrm{i}t)^{p+|l|+1}}L_p^{|l|}\left(\frac{2r^2}{1+t^2}\right)\exp\left[-\frac{r^2}{1-\mathrm{i}t}\right],
\end{aligned} \tag{9}$$

were $x = r\omega_0\cos\phi$, $y = r\omega_0\sin\phi$, $z = z_R t$; $z_R = \pi\omega_0^2/\lambda$ is the Rayleigh range, $L_p^{|l|}$ represents the generalized Laguerre polynomials with the parameters l and p being the azimuthal and the radial mode indices respectively; ω_0 is the beam waist and

$$\mathcal{N} = \left[\frac{p!2^{|l|+1}}{\pi(p+|l|)!}\right]^{1/2} \tag{10}$$

is the normalization constant. The mode is propagating in the z-direction and we assume that the photons are monochromatic and propagate paraxially.

Initially, the source generates two photons in the Bell state

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|q\rangle_A|\bar{q}\rangle_B + |\bar{q}\rangle_A|q\rangle_B\right), \tag{11}$$

where $q = 1$ and $\bar{q} \equiv -1$. When a photon in a given LG_{lp}-mode is propagated in the turbulent atmosphere, the state of the photon will be scattered into all the other OAM modes. In other words, the state of the photon after propagation is a superposition of all the modes with different l and p values. Since we are only interested in qubits in this work, we only extract the information contained in the modes where $l = \pm 1$ and $p = 0$. Hence, we neglect the coupling into higher modes. By doing so, we obtain a density matrix that is not normalized. The state of photon A or B changes as follows after it goes through a phase screen:

$$\begin{aligned}
|q\rangle_A \;&\rightarrow\; a_q|q\rangle_A + a_{\bar{q}}|\bar{q}\rangle_A \\
|\bar{q}\rangle_A \;&\rightarrow\; c_q|q\rangle_A + c_{\bar{q}}|\bar{q}\rangle_A \\
|\bar{q}\rangle_B \;&\rightarrow\; b_q|q\rangle_B + b_{\bar{q}}|\bar{q}\rangle_B \\
|q\rangle_B \;&\rightarrow\; d_q|q\rangle_B + d_{\bar{q}}|\bar{q}\rangle_B,
\end{aligned} \tag{12}$$

where a_q, $a_{\bar{q}}$ etc. are complex coefficients that are calculated by the inner products $a_q = \langle q|\psi\rangle_A$, $a_{\bar{q}} = \langle \bar{q}|\psi\rangle_A$ etc. and $|\psi\rangle_A$ is the state of photon A after it has propagated through the turbulence.

After propagation through turbulence, the initial state given by Eq. (11) will be transformed into

$$\begin{aligned}
|\Psi_{AB}\rangle \rightarrow |\Psi_{AB}^{out}\rangle \;&=\; C_1|q\rangle_A|q\rangle_B + C_2|q\rangle_A|\bar{q}\rangle_B \\
&\quad +C_3|\bar{q}\rangle_A|q\rangle_B + C_4|\bar{q}\rangle_A|\bar{q}\rangle_B,
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
C_1 \;&=\; \frac{1}{\sqrt{2}}\left(a_q b_q + c_q d_q\right) \\
C_2 \;&=\; \frac{1}{\sqrt{2}}\left(a_q b_{\bar{q}} + c_q d_{\bar{q}}\right) \\
C_3 \;&=\; \frac{1}{\sqrt{2}}\left(a_{\bar{q}} b_q + c_{\bar{q}} d_q\right) \\
C_4 \;&=\; \frac{1}{\sqrt{2}}\left(a_{\bar{q}} b_{\bar{q}} + c_{\bar{q}} d_{\bar{q}}\right).
\end{aligned} \tag{14}$$

To reconstruct the density matrix describing the state of the two qubits after each phase screen, a number $N = 1000$ of initial states are generated and sent through the phase screens. Each initial state is assumed to remain in a pure state as it propagate though the phase screens. After each phase screen, the average over all N

density matrices describing the initial states is calculated. For instance the density matrix of the system after the ith phase screen is given by

$$\rho_i = \frac{\sum_j^N |\Psi_j^i\rangle\langle\Psi_j^i|}{\text{Tr}\left[\sum_j^N |\Psi_j^i\rangle\langle\Psi_j^i|\right]}, \tag{15}$$

where $|\Psi_j^i\rangle$ represents the state of the jth initial state after the ith phase screen. We calculate the density matrix by averaging over all the initial states. This averaging is needed because the environment is random and we can only get information on its state by a statistical analysis. The averaging has the effects of removing the randomness of the environment. The density matrix describing the state of the qubits becomes mixed after the averaging.

The concurrence, which is used as a measure of entanglement, is given by[9]

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}. \tag{16}$$

The λ_i are the eigenvalues, in decreasing order, of the Hermitian matrix

$$R = \rho\tilde{\rho}, \tag{17}$$

where $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ is known as the spin-flip state and

$$\sigma_y = \begin{bmatrix} 0 & -\text{i} \\ \text{i} & 0 \end{bmatrix} \tag{18}$$

is the Pauli y matrix.

3. NUMERICAL SIMULATION

3.1 Numerical procedure

Since we want to simulate the evolution of quantum entanglement between two photons initially in the Bell state $|\Psi_{AB}\rangle$ given in Eq. (11), we start with four optical fields corresponding to the different possibilities in Eq. (11) (photon A in $|q\rangle$, photon A in $|\bar{q}\rangle$, photon B in $|q\rangle$, photon B in $|\bar{q}\rangle$). The four optical fields are then propagated in each run of the simulation. These fields are input with 256×256 arrays of complex numbers representing the mode. The function representing the mode initially is given by Eq. (9), where the radial index p is always set to zero and the azimuthal index l is set to ± 1. The sample spacing d is appropriately chosen to avoid aliasing ($d = \omega_0/20$ in all the results except when $K = 10$ and $K = 30$, in which cases $d = \omega_0/10$).

The next step in the simulation after initializing the optical fields is the phase modulation. That is, we multiply the array representing the field with the array representing the phase screen computed with Eq. (6). After that, the beams are propagated over the distance Δz through free space. The distance Δz is always chosen to be $1/100$ of the total propagation distance, and the total propagation distance varies depending on the values of the parameters. The phase modulation followed by the free-space propagation are repeated for all the phase screens using the distorted beams from the previous phase screens.

After each propagation step, the density matrix of the resulting quantum states is reconstructed by extracting the coefficients of the different modes from the four beam profiles at that point and combining these coefficients into the expression for the states according to Eq. (13).

Each phase screen represents a specific instance of the turbulent atmosphere. Thus the density matrix obtained after each phase screen is that of a pure state. But since the turbulence is random, we repeat the propagation process for $N = 1000$ initial states. Each initial state goes through a random phase screen representing a different instance of the turbulent atmosphere. The average of all the N density matrices is computed and normalized after each propagation step as expressed in Eq. (15). This averaging is necessary to eliminate the randomness of the turbulence; the resultant density matrix represents a mixed state.
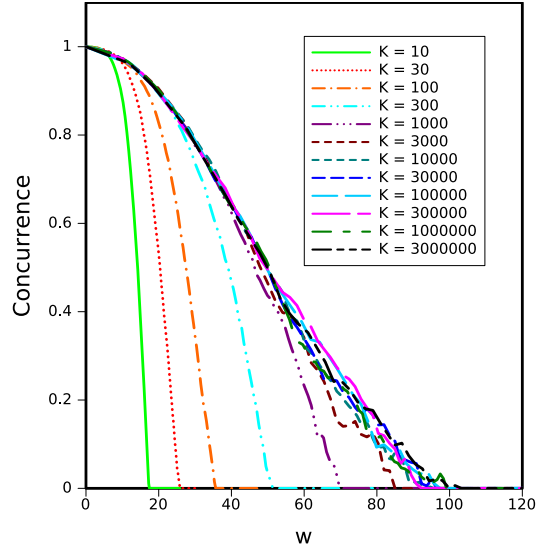
Figure 2. Plot of the concurrence against the quantity $w = \omega_0/r_0$, where ω_0 is the beam radius and r_0 is the fried parameter, for different values of K.
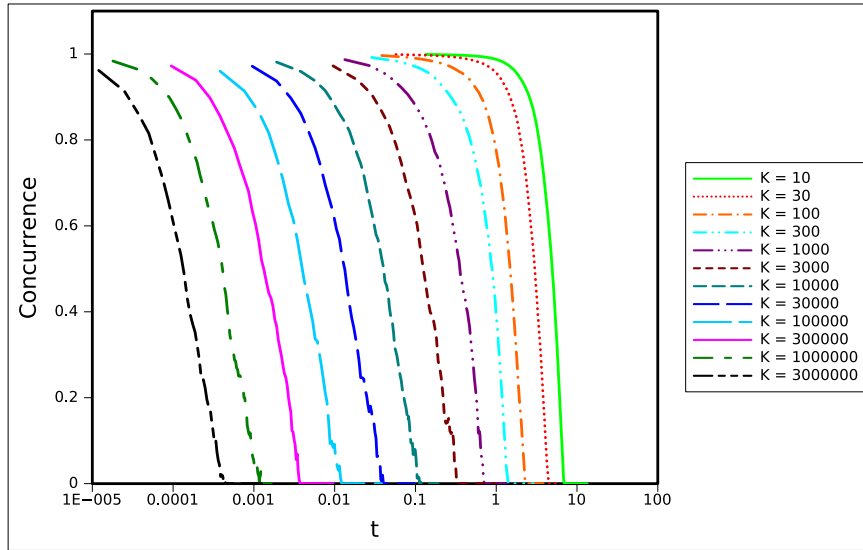

Figure 3. Plot of the concurrence against the quantity $t = z/z_R$ where z is the propagation distance and z_R the Rayleigh range for different values of K.

3.2 Results and discussions

The main aim of this work is to study how the parameters (the beam radius, the wavelength and the refractive index structure function parameter) influence the evolution of quantum entanglement as the beam propagate in a turbulent atmosphere. It is customary to combine these parameters into the Fried parameter[14] which, for plane waves, is given by

	Fig.5(a)	Fig.5(b)	Fig.5(c)	Fig.5(d)	Fig.4(a)	Fig.4(b)	Fig.4(c)	Fig.4(d)
ω_0(cm)	4	10	20	40	4	5	10	40
λ(nm)	632.8	4177.7	452.38	489.88	839.2	600	554.1	649.8
C_n^2(m$^{-2/3}$)	10^{-13}	10^{-14}	10^{-16}	10^{-17}	10^{-13}	1.6125×10^{-14}	10^{-15}	10^{-17}

Table 1. parameters used in Fig.4 and 5.

$$r_0 = 0.185 \left(\frac{\lambda^2}{C_n^2 L} \right)^{\frac{3}{5}},$$

where λ is the wavelength, L is the propagation distance and C_n^2 is the refractive index structure function parameter which determine the strength of the turbulence. In this case, one then studies the evolution of the concurrence with respect to the dimensionless quantity $w = \omega_0/r_0$. According to the analytical analysis done by one of the authors,[15] one can alternatively combine the parameters into the dimensionless quantity

$$K = \frac{C_n^2 \omega_0^{11/3} \pi^3}{\lambda^3}. \tag{19}$$

One can then study the evolution of the concurrence against the quantity $t = z/z_R$ where z is the propagation distance and z_R is the Rayleigh range. Both of these approaches are considered in this work. The dimensionless quantities w and K are related through the equation $w = Kt$. Fig. 2 shows the plots of the concurrence versus w for different values of K, and Fig. 3 show the plots of the concurrence against t for the same values of K used in Fig. 2. It is interesting to note from Fig. 3 that the bigger the value of K, the quicker the concurrence decays to zero with respect to t. The opposite is observed in Fig. 2, that is, the bigger the value of K, the longer the concurrence takes to decay to zero with respect to w. This can be explained by the relationship between w and K. We also see from Fig. 2 that as K increases, the curves of the concurrence against w seem to approach a limiting curve. The curves corresponding to $K = 3000$ to 3000000 more or less coincide with each others. This suggests that there is a value of K beyond which the evolution of the concurrence with respect to w is independent of the values of the dimension parameters (λ, ω_0, and C_n^2).

It was reported in Ref. 3 that the evolution of entanglement depends only on the dimensionless quantity $w = \omega_0/r_0$. Our results suggest that the previous statement is true only if the overall propagation distance is shorter than the Rayleigh range. As we can see from Fig. 3, the curves coinciding with one another in Fig. 2 (K values between 3000 - 3000000) all reach zero at a value of $t \lesssim 0.3$ when plotted against t. This is well in agreement with the theoretical analysis done by one of the authors,[15] where he deduced that the evolution of the concurrence depended only on w if $t \lesssim 1/3$.

We also observed that the curves of the concurrence corresponding to the same value of K are identical regardless of the values of the different dimension parameters. This can be seen in Fig. 4 and 5, where different values of the parameters are chosen, as shown in Table 1. These sets of parameters all lead to the same value of K (39.27 in Fig. 4 and 91.6 in Fig. 5).
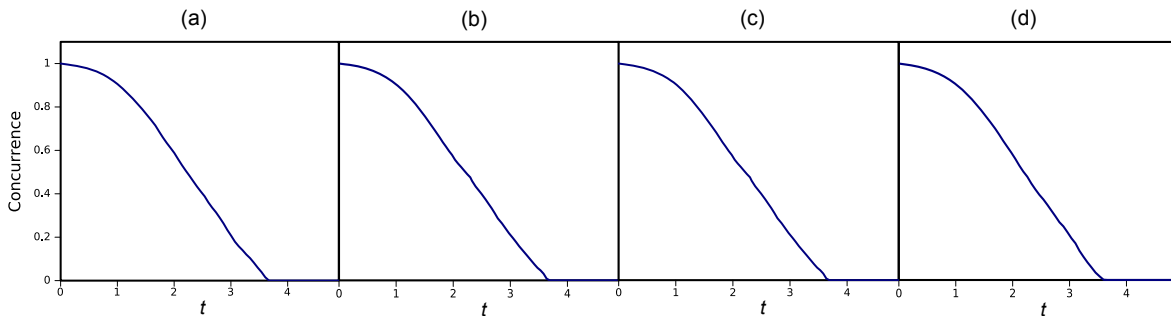

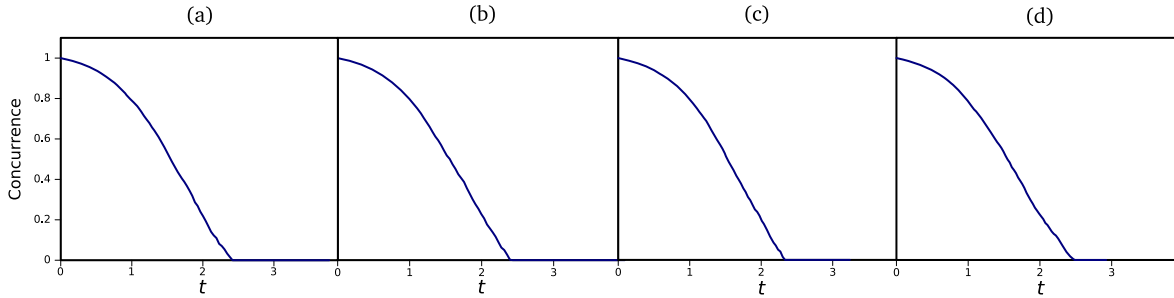Figure 4. Plot of the concurrence against t for K = 39.27.

Figure 5. Plot of the concurrence against t for K = 91.6.

4. CONCLUSION

We presented a numerical study of the evolution of quantum OAM entanglement between a pair of photons propagating in a turbulent atmosphere. Different values of the parameters were used. We have shown that the evolution of the concurrence depends only on a dimensionless quantity. In other words, regardless of the values of the different dimension parameters (ω_0, λ or C_n^2), all the curves of the concurrence corresponding to the same value of K are identical.

It was reported in Ref. 3 that the evolution of the concurrence depended only on the dimensionless quantity $w = \omega_0/r_0$. We show that this is true only in the case where the overall propagation distance is at least three times shorter than the Rayleigh range. The same conclusion was reached in Ref. 15. In the case where the propagation distance is shorter than the Rayleigh range, our results agree qualitatively with those presented in Ref. 3 with the difference that the concurrence takes a longer time to decay to zero when plotted against w in our simulation.

ACKNOWLEDGMENTS

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