

# On the Trade-off Between Mainlobe Width and Peak Sidelobe Level of Mismatched Pulse Compression Filters for Linear Chirp Waveforms

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**Abstract—** In [1] the authors introduced a technique for generating mismatched pulse compression filters for linear frequency chirp signals. The technique minimizes the sum of the pulse compression sidelobes in an  $L_p$ -norm sense. It was shown that constant, extremely low sidelobe levels (better than 60 dB) can be achieved for minimal mismatch loss but with some broadening of the compression peak. This paper investigates the tradeoff between the sidelobe level and the mainlobe width for mismatched filters designed using  $L_p$ -norm techniques.

## I. INTRODUCTION

Pulse compression gives radar designers the ability to obtain sufficient energy on a target for target detection without decreasing the range resolution of the system or resorting to the use of very high power transmitters. Pulse compression thus permits the use of lower power transmitters with longer pulse lengths to maintain the energy content of a pulse. A matched filter is used on reception to maximize the signal to noise ratio of the received signal [3]. The actual transmitted waveforms are chosen so as to have an autocorrelation function with a narrow peak at zero time shift and sidelobe values as low as possible at all other times. The sidelobes have the undesirable effect of masking smaller targets in close proximity to larger targets, such as clutter returns.

No direct design techniques exist for the generation of pulse compression waveforms with optimally low sidelobe levels. This has resulted in several approaches to the problem of designing “good” pulse compression waveforms. Early pulse compression systems were based on linear frequency modulated waveforms [4], [5] so several techniques have been developed to reduce the sidelobe levels of this broad class of pulse compression waveforms. These include amplitude windowing of the signal and the dual thereof which is frequency windowing. Notably, De Witte and Griffiths [6] achieved sidelobe levels of approximately 70 dB by using non-linear frequency chirp waveforms, with mismatched filters on receive. Optimal binary phase shift codes have been found by means of exhaustive search techniques [7] – [10]. Searches for good quadriphase codes have been reported in

[11] and [12]. Gartz [13] and Nunn [14] have addressed the search for polyphase codes. Code searches are constrained by the computational complexity of the search process, which limits the lengths of codes which can be found using this technique. This has led some authors to develop techniques for constructing codes which have close to optimal sidelobe levels. The Frank codes [15] and P(n,k) codes developed by Felhauer [16] are well known codes in this category. Recently, a technique, based on the Zak transform, was derived by Gladkova [17] for the generation of non-linear frequency modulated waveforms. Post-processing of the pulse compressor output such as sidelobe cancellation [18] and sidelobe smoothing [19] have also been reported. The use of long mismatched filters for binary codes was investigated by Levanon [20]. These filters were up to three times the length of the original code. Envelope constrained filter design techniques [21] have also been applied to the design of mismatched filters with low sidelobes.

Making use of pulse compression filters which have been deliberately mismatched to reduce the sidelobe levels has the detrimental effects of a loss in signal to noise ratio as well as broadening of the compression peak as reported in [1] and [2]. An extension of this sidelobe reduction technique, which allows the designer to control the mainlobe width, is described in the following section.

## II. REDUCTION OF SIDELOBE LEVELS BY MINIMIZATION OF A NORM APPLIED TO THE INTEGRATED SIDELOBE LEVEL

The mismatched pulse compression filter can be designed by minimizing the sum of the squared magnitudes of the complex sidelobe values ( $L_2$ -norm). This leads to the minimization of the energy in the sidelobes, whereas minimization of the  $L_\infty$ -norm will minimize the peak sidelobe level. The  $L_\infty$ -norm is not a well behaved function, so  $L_p$ -norms with  $p = 2P$ ,  $1 \leq P \leq 200$ , were used in [1] and the solutions were found by means of numerical techniques. This section extends the techniques in [1] in such a way that the amplitude of the samples to either side of the peak sample can be specified. This allows the width of the peak to be

specified indirectly, and then the best sidelobe level response is found for the given peak width. This problem is formulated as an optimisation problem with three constraints, and is solved using Lagrange multipliers.

Given the discrete time transmit sequence  $\{a_n\}$  and filter coefficients  $\{x_n\}$ , the output of the filter is given by

$$b_i = \sum_k a_{i+1-k} x_k. \quad (1)$$

For the matched filter case  $x_n = a_{N-n}^*$ . The output of the pulse compression filter can also be written in matrix form as

$$\mathbf{b} = \mathbf{A}_F \mathbf{x}, \quad (2)$$

where

$$\mathbf{b} = [b_1 \quad b_2 \quad \cdots \quad b_{2N-1}]^T, \quad (3)$$

$$\mathbf{x} = [x_1 \quad x_2 \quad \cdots \quad x_N]^T \quad (4)$$

and

$$\mathbf{A}_F = \begin{bmatrix} a_1 & a_2 & \cdots & a_N & 0 & 0 & \cdots & 0 \\ 0 & a_1 & a_2 & \cdots & a_N & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_1 & a_2 & \cdots & a_N & 0 \\ 0 & \cdots & 0 & 0 & a_1 & a_2 & \cdots & a_N \end{bmatrix}^T. \quad (5)$$

The above formulation leads to the following convenient expression for the sum-of-squares of the convolution sequence:

$$\mathbf{b}^H \mathbf{b} = \|b_1\|^2 + \|b_2\|^2 + \cdots + \|b_{2N-1}\|^2. \quad (6)$$

The sidelobe measure function can now be formulated by defining a new matrix  $\mathbf{A}$  which excludes the rows in  $\mathbf{A}_F$  which produce the compression peak and its two adjacent samples.

The  $L_2$ -norm cost function to be minimized is given by

$$f(\mathbf{x}) = \mathbf{x}^H \mathbf{C} \mathbf{x} \quad (7)$$

where

$$\mathbf{C} = \mathbf{A}^H \mathbf{A} \quad (8)$$

The method of Lagrange multipliers [22] is used to find a solution for  $\mathbf{x}$  that will minimize the sidelobe measure cost function while satisfying the constraint that a pulse compression peak with amplitude  $b_{peak}$  must be produced. In addition to this constraint, two more are added, one for each of the samples adjacent to the peak sample. The set of constraints is thus given by

$$g_1(\mathbf{x}) = \mathbf{a}_1 \mathbf{x} - b_1 = 0 \quad (9)$$

$$g_2(\mathbf{x}) = \mathbf{a}_2 \mathbf{x} - b_{peak} = 0 \quad (10)$$

$$g_3(\mathbf{x}) = \mathbf{a}_3 \mathbf{x} - b_2 = 0 \quad (11)$$

where

$$\mathbf{a}_1 = [a_{n-1} \quad a_{n-2} \quad \cdots \quad a_1 \quad 0] \quad (12)$$

$$\mathbf{a}_2 = [a_n \quad a_{n-1} \quad \cdots \quad a_1] \quad (13)$$

$$\mathbf{a}_3 = [0 \quad a_n \quad a_{n-1} \quad \cdots \quad a_2] \quad (14)$$

and  $b_1$  and  $b_2$  are the required values for the samples adjacent to the peak sample.

The Lagrange multiplier formulation with multiple constraints is given by

$$\frac{d}{d\mathbf{x}^*} (f(\mathbf{x})) + \sum_{k=1}^K \frac{d}{d\mathbf{x}^*} (\text{Re}\{\lambda_k^* g_k(\mathbf{x})\}) = \mathbf{0}. \quad (15)$$

If the derivatives in (15) are evaluated for each of the terms, the following results per term can be derived:

$$\frac{d}{d\mathbf{x}^*} (f(\mathbf{x})) = \mathbf{C} \mathbf{x} \quad (16)$$

$$\frac{d}{d\mathbf{x}^*} (\text{Re}\{\lambda_k^* g_k(\mathbf{x})\}) = \frac{1}{2} \lambda_k \mathbf{a}_k^H. \quad (17)$$

Substitution of these results into (15) gives

$$\mathbf{C} \mathbf{x} + \frac{1}{2} \lambda_1 \mathbf{a}_1^H + \frac{1}{2} \lambda_2 \mathbf{a}_2^H + \frac{1}{2} \lambda_3 \mathbf{a}_3^H = \mathbf{0}. \quad (18)$$

The system of equations to solve is thus given by

$$\mathbf{C} \mathbf{x} + \frac{1}{2} \lambda_1 \mathbf{a}_1^H + \frac{1}{2} \lambda_2 \mathbf{a}_2^H + \frac{1}{2} \lambda_3 \mathbf{a}_3^H = \mathbf{0} \quad (19)$$

$$\mathbf{a}_1 \mathbf{x} = b_1 \quad (20)$$

$$\mathbf{a}_2 \mathbf{x} = b_{peak} \quad (21)$$

$$\mathbf{a}_3 \mathbf{x} = b_2. \quad (22)$$

Solving (19) for  $\mathbf{x}$  gives

$$\mathbf{x} = -\frac{1}{2} \mathbf{C}^{-1} (\lambda_1 \mathbf{a}_1^H + \lambda_2 \mathbf{a}_2^H + \lambda_3 \mathbf{a}_3^H). \quad (23)$$

Pre-multiplication by  $\mathbf{a}_1$  and substitution of (20) allows  $b_1$  to be found as:

$$b_1 = -\frac{1}{2} \mathbf{a}_1 \mathbf{C}^{-1} (\lambda_1 \mathbf{a}_1^H + \lambda_2 \mathbf{a}_2^H + \lambda_3 \mathbf{a}_3^H). \quad (24)$$

For the rest of the derivation it is important to note that terms of the form  $\mathbf{a}_x \mathbf{a}_y^H$  and  $\mathbf{a}_x \mathbf{C}^{-1} \mathbf{a}_y^H$  are complex scalars. It is thus possible to make the following notational change

$$c_{xy} = \mathbf{a}_x \mathbf{C}^{-1} \mathbf{a}_y^H \quad (25)$$

to simplify the derivation. Using this notation,  $b_1$  can be written as

$$b_1 = -\frac{1}{2} \lambda_1 c_{11} - \frac{1}{2} \lambda_2 c_{12} - \frac{1}{2} \lambda_3 c_{13}, \quad (26)$$

which in turn can be solved for  $\lambda_1$  to give

$$\lambda_1 = \frac{-(2b_1 + \lambda_2 c_{12} + \lambda_3 c_{13})}{c_{11}}. \quad (27)$$

This result can be substituted into (23) to give

$$\mathbf{x} = -\frac{1}{2} \mathbf{C}^{-1} \left( -\frac{2b_1 + \lambda_2 c_{12} + \lambda_3 c_{13}}{c_{11}} \mathbf{a}_1^H + \lambda_2 \mathbf{a}_2^H + \lambda_3 \mathbf{a}_3^H \right) \quad (28)$$

This process can be repeated for  $\lambda_2$  and  $\lambda_3$  to obtain

$$\lambda_2 = \frac{2(b_{peak} c_{11} - b_1 c_{21})}{c_{12} c_{21} - c_{22} c_{11}} - \left( \frac{c_{13} c_{21} - c_{11} c_{23}}{c_{12} c_{21} - c_{22} c_{11}} \right) \lambda_3 \quad (29)$$

$$\lambda_3 = 2 \frac{(b_2 c_{11} - b_1 c_{31})(c_{12} c_{21} - c_{22} c_{11}) - (c_{12} c_{31} - c_{11} c_{32})(b_{peak} c_{11} - b_1 c_{21})}{(c_{11} c_{32} - c_{12} c_{31})(c_{13} c_{21} - c_{11} c_{23}) + (c_{13} c_{31} - c_{11} c_{33})(c_{12} c_{21} - c_{22} c_{11})} \quad (30)$$

This leads to the following expression for  $\mathbf{x}$ :

$$\mathbf{x} = \mathbf{C}^{-1} \left\{ \begin{aligned} & \frac{b_1}{c_{11}} \mathbf{a}_1^H + \left( \frac{c_{12}}{c_{11}} \mathbf{a}_1^H - \mathbf{a}_2^H \right) \left( \frac{b_{peak} c_{11} - b_1 c_{21}}{c_{12} c_{21} - c_{22} c_{11}} \right) + \\ & \left[ \left( \frac{c_{13}}{c_{11}} \mathbf{a}_1^H - \mathbf{a}_3^H \right) - \left( \frac{c_{12}}{c_{11}} \mathbf{a}_1^H - \mathbf{a}_2^H \right) \left( \frac{c_{13} c_{21} - c_{11} c_{23}}{c_{12} c_{21} - c_{22} c_{11}} \right) \right] \times \\ & \times \left[ \frac{(b_2 c_{11} - b_1 c_{31})(c_{12} c_{21} - c_{22} c_{11}) - (c_{12} c_{31} - c_{11} c_{32})(b_{peak} c_{11} - b_1 c_{21})}{(c_{11} c_{32} - c_{12} c_{31})(c_{13} c_{21} - c_{11} c_{23}) + (c_{13} c_{31} - c_{11} c_{33})(c_{12} c_{21} - c_{22} c_{11})} \right] \end{aligned} \right\} \quad (31)$$

Equation (31) can be implemented numerically to calculate the mismatched filter for the  $L_2$ -norm case.

For the  $L_{2P}$ -norm case the cost function is given by

$$f(\mathbf{x}) = \left( \sum_{i=1}^{2N-1} \left[ \mathbf{x}^H \mathbf{C}_i \mathbf{x} \right]^p \right)^{\frac{1}{2P}}, \quad (32)$$

which has to be substituted into (15) and then solved simultaneously with the three constraints given by (9), (10) and (11). The derivative of the cost function was derived in [1] and is given by

$$\frac{d}{d\mathbf{x}^*} (f(\mathbf{x})) = \frac{\sum_{i=1}^{2N-1} \left( \left[ \mathbf{x}^H \mathbf{C}_i \mathbf{x} \right]^{p-1} \mathbf{C}_i \mathbf{x} \right)}{\left( \sum_{i=1}^{2N-1} \left( \left[ \mathbf{x}^H \mathbf{C}_i \mathbf{x} \right]^p \right) \right)^{1-\frac{1}{2P}}}. \quad (33)$$

The final version of (15), along with the three constraints has to be solved numerically for the  $L_{2P}$ -norm case.

To reduce the sidelobe level further, the pulse compression filter was symmetrically extended in time to be longer than the transmitted pulse, as described in [1]. These extra filter coefficients are referred to as additional non-zero coefficients (ANZC).

### III. RESULTS

To illustrate the results achieved with the technique derived above, several examples are presented in this section. The first example (Fig. 1) shows the output of the mismatched filter for a time-bandwidth product (TBWP) of 50 and a mismatched filter length of 150 samples. The samples to either side of the peak have been constrained to be 7 and 10 dB below the peak. The right hand plot is a zoomed view of four samples to either side of the peak. This plot compares two selected results for the  $L_2$ -norm and the  $L_{200}$ -norm.

Fig. 2 shows a plot of the results for the  $L_2$ -norm case, where the adjacent samples were constrained at 5, 10, 20, and 50 dB below the peak. In the right hand plot the peak sample and the four samples to the right of the peak have been plotted for clarity. Fig. 3 is a repeat of the same parameters as Fig. 2, except that the mismatched filters have been designed by minimization of the  $L_{100}$ -norm.

From Fig. 1, Fig. 2 and Fig. 3 it can clearly be seen that the sidelobe level increases as the samples adjacent to the peak are reduced. The optimization process is thus trading mainlobe width for sidelobe level.

From the results presented in Fig. 2 and Fig. 3 it appears that the mismatch loss (SNR loss at the peak) decreases as the adjacent sample is decreased.

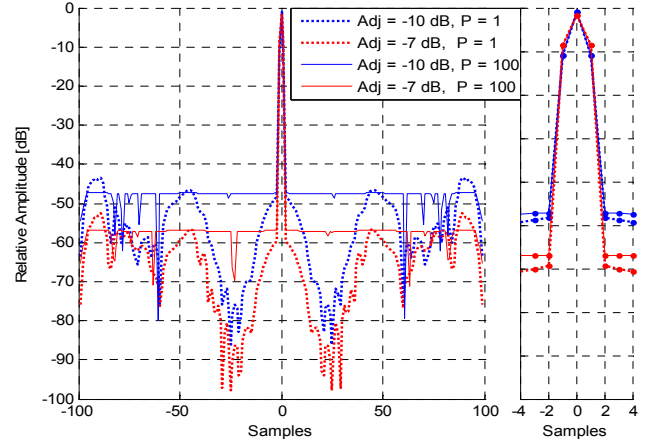


Fig. 1 Examples of  $L_2$  and  $L_{200}$  mismatched filters with TBWP=50, ANZC=50 and the samples adjacent to the peak constrained to be 10 dB and 7 dB below the peak.

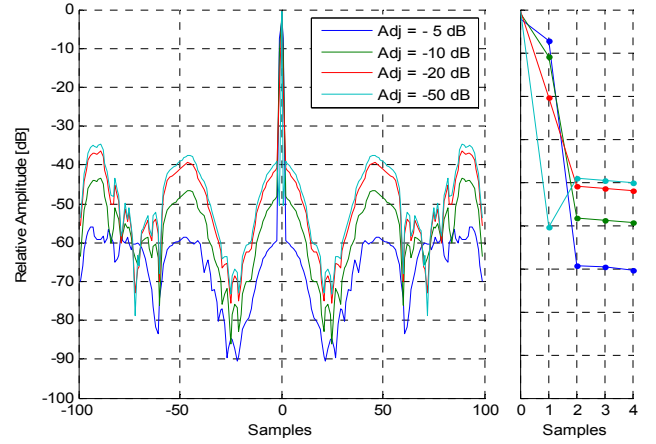


Fig. 2 Examples of  $L_2$  mismatched filters with TBWP=50, ANZC=50 and the samples adjacent to the peak constrained to be 5, 10, 20 and 50 dB below the peak.

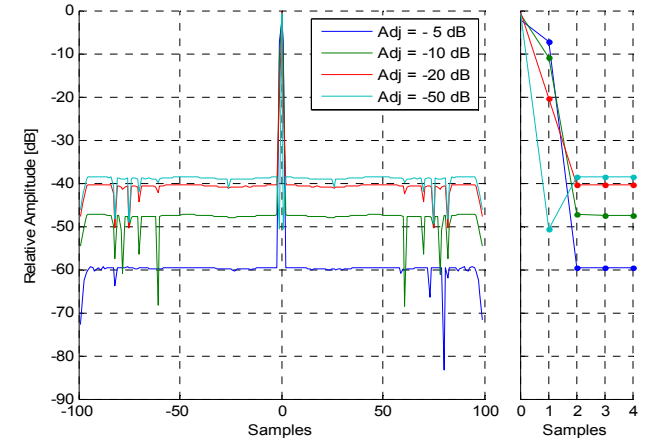


Fig. 3 Examples of  $L_{100}$  mismatched filters with TBWP=50, ANZC=50 and the samples adjacent to the peak constrained to be 5, 10, 20 and 50 dB below the peak.

To investigate this phenomenon, the relative amplitude of the adjacent samples were varied from 1 dB to 70 dB. The results are presented in Fig. 4.

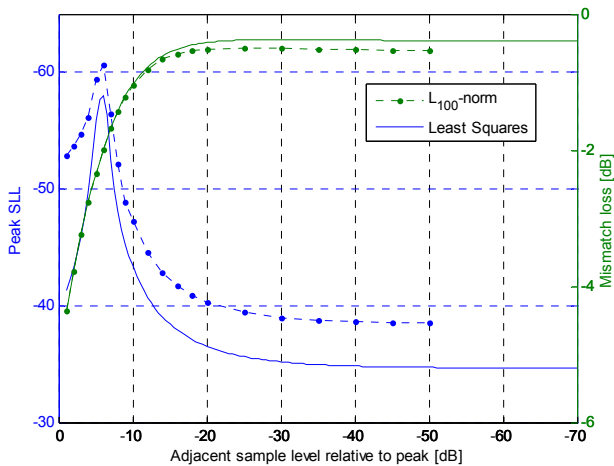


Fig. 4 Highest sidelobe level to peak ratio and mismatch loss as a function of the adjacent sample level for the  $L_2$  and  $L_{100}$  cases, with TBWP=50, ANZC=50.

Fig. 4 shows the effect of the additional constraints on the peak SLL and the mismatch loss for the least squares case and the  $L_{100}$ -norm case. The lowest value of mismatch loss occurs at the point where the adjacent samples are constrained to 29 dB for the least squares case and 25 dB for the  $L_{100}$ -norm case. The value of the mismatch loss at this point is 0.38 dB and 0.50 dB respectively. The lowest value of the peak SLL occurs when the adjacent samples are constrained to 6 dB in both cases. The mismatch losses at this point are approximately 1.9 dB for both cases, which is a relatively high loss to incur.

The figure shows that the higher norm produces better sidelobe levels in exchange for a slight degradation in mismatch loss than for the least squares case. In general the adjacent sample will probably not be set to values below 20 dB below the peak, unless the radar designer would like to optimize a parameter such as range estimation accuracy without considering the sidelobe levels or the mismatch loss.

#### IV. CONCLUSION

From Fig. 1 to Fig. 3 it can be seen that the samples adjacent to the peak sample have been successfully controlled by the additional constraints in the optimization problem.

Fig. 4 clearly shows the trade-off between peak SLL, mismatch loss and peak width (which is directly related to the adjacent sample level). The present technique therefore allows the radar designer to optimize the radar pulse compression for a specific application by altering the pulse compression receive filter, while the transmission waveform remains a linear chirp. The raw range line could even be processed using two different pulse compression filters in parallel, each feeding a different process within the radar, where the processes have different requirements on the pulse compression sub-system.

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