A Method of Applying Single Higher Order Polynomial Basis Function over Multiple Domains

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Abstract— A novel method has been devised where one set of higher order polynomial-based basis functions can be applied over several wire segments, thus permitting to decouple the number of unknowns from the number of segments, and so from the geometrical approximation accuracy. The method extends the current state of art from using the composite piecewise uniform, linear and sinusoidal basis and testing functions onto polynomials. The method has been derived within the framework of a method of moments (MoM) with higher-order polynomial basis functions, and applied to a surface form of the electrical field integral equation, under thin wire approximation. The main advantage of the proposed method is in permitting to reduce the required number of unknowns when modeling curved structures and structures including electrically small features. Derivation of the computational complexity in terms of floating point operations (FLOP) showed a possible speed gain nearly an order of the number of unknowns of direct MoM.

1. INTRODUCTION

The process of numerical electromagnetic modeling may be split into several stages [1], including geometry representation/modeling, representation/modeling of the current distribution, solution of the equations, and computation of the radiation and other parameters. A dominant majority of existing theoretical frameworks and available codes are based on approximating the geometry in a piecewise linear manner. This is sufficient for many practical geometrical structures having small localized curvatures but may be inefficient for structures with a large quantity of curved surfaces, like reflector antennas, fuselage of aircrafts and cars etc. The inefficiency usually manifests itself in the number of unknowns used by a method per unit of length or area. If the number of unknowns per unit of length/area is much higher than would have been otherwise required for a straight/flat surface, this is usually because an accurate geometrical approximation demanded a higher number of geometrical segments and associated basis functions than it is necessary to approximate the current distribution.

One possible way to improve efficiency is to use curved geometrical segments [2–4, 10]. This is still a rarely used technique and requires a substantial investment in analytical work. In addition, it cannot be used with or on top of existing codes directly or easily.

This paper presents a new higher-order method permitting to decouple the number of unknowns (i.e., current approximation accuracy) from the number of geometrical segments, and thus also from the geometrical approximation accuracy for curved structures. The method enables aggregation of multiple geometrical segments under the same set of basis/expansion functions. This permits to decouple the number of unknowns and current approximation accuracy from the number of geometrical segments and from the required accuracy of geometrical approximation accuracy. Aggregation of individual electrically small features also aims a reduction in the impedance matrix condition number discussed in [5, 6]. The grounds for this assumption are given in [7].

The method proposed extends the prior art based on the piecewise uniform [8], linear [5] and sinusoidal [9] basis and testing functions onto usage of polynomial basis functions [1]. The idea of aggregating and/or re-arranging the basis functions has been reflected in several different types of basis functions ranging from the use of splines [10], composite basis functions [8], macro basis functions and characteristic basis functions [11–13], to the initial steps into multiple domain basis functions [5, 6, 9].

The work introduces a method permitting to apply higher order basis functions [1] over multiple geometrical segments. The reason to develop this approach is to increase the efficiency of numerical modeling by reducing the number of basis functions required per unit of length or area. This follows an accepted understanding of higher efficiency of higher order polynomial basis functions [1, 10] compared to more traditional pulse and piecewise linear (low order) basis functions.

A special note needs to be made with regards to a comparison of the method proposed, to the characteristic basis functions [13]. The characteristic basis functions require to solve a set of

subdomain/local problems, thus permitting to find the problem-specific shape of basis functions first. This is then followed by using the obtained profiles of basis functions to obtain the global solution under the usual MoM framework. This technique shows high degree of efficiency and accuracy in solving problems. However, it is assumed that on simply-shaped geometries like large smoothly bent structures, the need to apply full MoM solution to each local problem will exceed the amount of resources required to aggregate the basis functions and determine the higher order solution. In addition, the characteristic basis functions are expected to require more work in integrating them with existing numerical MoM codes, compared to the approach proposed in this paper.

Another note needs to be made concerning the applicability of any "macro" type of techniques, including MDBF. Presence of geometrical elements near the domain of an MDBF which can couple to the elements described by this domain, may disturb the current distribution and fields described by the assumed shape of basis function and lead to incorrect results [14]. This is a task of setting a correct mesh and needs to be taken into account during the geometrical modeling stage, when traditional basis functions and/or MDBF are applied.

This paper is organized as follows. Section 2 of this paper provides the theoretical basis for the method proposed. It also includes some suggestions regarding optimizing the computations. The next section offers a discussion on the computational complexity of the method and gives a measure for selecting between the method proposed and a direct MoM solution.

2. THEORY OF THE MULTIPLE-DOMAIN BASIS FUNCTIONS

The technique is based on the following approximating equation [5, 6, 8]

$$\mathbf{I}_1 = \mathbf{M} \cdot \mathbf{I}_2$$

relating the unknowns in a traditional (original) formulation of the MoM to the unknowns in the new formulation [5, 6, 9]. In this equation, $\mathbf{I}_1$ is the column vector with N_1 original unknowns, which are the unknowns used in the direct solution with the traditional method of moments. The column vector $\mathbf{I}_2$ is the vector with N_2 new unknowns. The compression of the impedance matrix is based on N_2 being smaller than N_1. The matrix $\mathbf{M}$ of size N_1 by N_2, relates the two sets of unknowns, and is herein referred to as a *compression matrix*.

The method discussed in this work focuses on establishing the matrix $\mathbf{M}$, when both the original (sub-domain) and new (composite) sets of basis functions are hierarchical polynomials. For simplicity, the derivations are done for a single continuous wire represented by a multiplicity of shorter wire segments. The derivations can be readily expanded onto an arbitrary combination of the wires or quadrilaterals.

It is assumed that each original basis function is written as $a_k \cdot (x - x_{c1})^k$, where x_c and x_{c1} are in a coordinate system common for all the wire segments composing the wire. A composite basis function (MDBF) is supposed to cover several wire segments. It is also assumed that each MDBF covers K wire segments (which is also the ratio of the number of wires to the number of chains). A similar form of $A_k \cdot (x - x_c)^k$ is assumed for the new compressed set of basis functions over the wire. The new co-ordinate system can be related to the original local co-ordinate systems by a simple shift in the coordinate.

The work shows that the matrix $\mathbf{M}$ can be written in the form

$$\mathbf{M} \equiv \mathbf{X} \cdot \mathbf{G} \cdot \mathbf{X}^{-1}.$$

Here, the matrix $\mathbf{X}$ relates the set of original basis functions, $\mathbf{BF}_{\text{old}}$, and their decomposition into terms of a polynomial, $\mathbf{P}_{\text{old}}$ as

$$\mathbf{BF}_{\text{old}} = \mathbf{X} \cdot \mathbf{P}_{\text{old}},$$

while the matrix $\mathbf{G}$ defines a conversion from the hierarchical polynomials defined in the original local coordinate system, $\mathbf{P}_{\text{old}}$, into the new local co-ordinate system, $\mathbf{P}_{\text{new}}$ (i.e., the relationship between the coefficients a_k and A_k):

$$\mathbf{P}_{\text{old}} = \mathbf{G} \cdot \mathbf{P}_{\text{new}}.$$

The elements of the matrices have been derived in the following manner described in detail in [6]. Assuming the relationship $[a_k] = \mathbf{G} \cdot [A_k]$, the elements of the matrix G can be found as $G_{ij} = C_{i-1}^{j-1} \Delta^{k-i}$, where the factor C_n^k is the binomial coefficient defined [15] as $C_n^k \equiv \binom{k}{n} \equiv \frac{k!}{n!(k-n)!}$. The matrix $\mathbf{G}$ is lower triangular.

The transformation between a basis polynomial set $\{x^i\}_i = 1, 2, \ldots, N$ represented as a column vector $[P]_{N\times 1}$, and the set of polynomial basis function represented as a column vector $[\mathrm{BF}]_{N\times 1}$ may be written in a matrix form as

$$[\mathrm{BF}] = [X] \cdot [P].$$

Specific entries of the matrix $\mathbf{X}$ depend on the choice of basis functions. Expanding this notation for nodal and singleton basis functions $\{N_i, S_j\}_{i=1,2;\, j=3,4,\ldots,N}$ defined in [1, 6], the above expression is written as

$$\underbrace{\begin{bmatrix} \frac{1}{2}(1-x) \\ \frac{1}{2}(1+x) \\ x^2-1 \\ x^3-x \\ x^4-1 \\ x^5-x \\ \vdots \end{bmatrix}}_{\mathbf{BF}} = \underbrace{\left[\begin{array}{cc|cccc|c} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \\ -1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 1 & 0 & 0 & \\ -1 & 0 & 0 & 0 & 1 & 0 & \\ 0 & -1 & 0 & 0 & 0 & 1 & \\ \vdots & \vdots & & \vdots & & & \ddots \end{array}\right]}_{\mathbf{X}} \cdot \underbrace{\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ \vdots \end{bmatrix}}_{\mathbf{P}}.$$

The matrixes $\mathbf{X}$ and $\mathbf{X}^{-1}$ do not need to be computed and can be stored to accelerate computations.

3. COMPUTATIONAL COMPLEXITY

The computational cost of the process has been estimated in terms of the floating point operations (FLOPs). This was done by adding up all the FLOPs required to establish the compression matrix, compress the system, solve it, and compute the original unknowns. The result was divided by the number of FLOPs required to obtain a solution via a direct MoM (assuming that the impedance matrix for the direct MoM had been filled in). Considering optimizations like storing the precomputed matrices $\mathbf{X}$ and $\mathbf{X}^{-1}$, the method proposed may provide a relative performance gain (speed up) of up to

$$K^3 \cdot (1 - 3/n_b) - 4K^7/N, \quad N \to \infty,$$

that is relative to the traditional MoM. Here the symbol n_b stands for the number of polynomial basis functions assigned per each geometrical segment, and N is the number of unknowns in the original uncompressed system due to the direct MoM.

From the last expression, it is clear that the number of basis functions assigned per each segment should be maximized in order to improve the efficiency of the method (i.e., use the highest possible order of basis functions).

Also, assuming that n_b and N are large, and analyzing the performance gain expression for a maximum gain, it is possible to show that the maximum relative performance gain with respect to a direct MoM is

$$0.107N^{3/4}(1 - 3/n_b)^{7/4} \tag{1}$$

This means that the method proposed can offer nearly an order of magnitude, $O(N^{3/4})$, advantage in performance, when large problems needs to be solved. A set of curves for the performance gain is shown in Figure 1. The figure indicates that the performance gain can be less than unity, making the method slower than the direct MoM is.

By setting the expression (1) to unity and expressing N via the remaining coefficients and parameter n_b, it is possible to show that the number of unknowns corresponding to the unity gain, N_1, is

$$N_1 = 19.7/(1 - 3/n_b)^{7/3}.$$

This expression can be used to determine whether it is better to use a direct MoM or the method proposed in this paper (if N_1 is smaller than N, then it is advantageous to apply the proposed method).

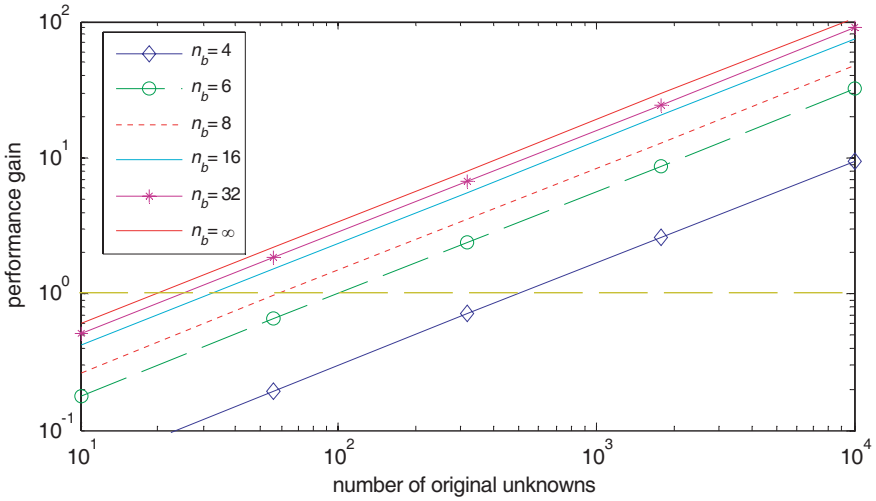

Figure 1. Maximum performance (relative acceleration with respect to the direct MoM) gain versus number of unknowns in the direct MoM; n_b stands for the number of basis functions assigned per geometrical segment.

4. CONCLUSION

A new higher-order method devised for efficient modeling of curved structures has been introduced. Some practical considerations for acceleration of computations related to the method are discussed. An estimate for the computational complexity of the method in terms of the count of floating-point operations (FLOP) has been given. It has been shown that the speed up of up to $O(N^{3/4})$ is possible.

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