Rotating structures and Bryan's effect

Stephan V. Joubert,[a)] Michael Y. Shatalov,[b)] and Temple H. Fay
Department of Mathematics and Statistics, Tshwane University of Technology, Private Bag x680, 0001 Pretoria, Gauteng, South Africa

(Received 14 July 2008; accepted 4 February 2009)

In 1890 Bryan observed that when a vibrating structure is rotated the vibrating pattern rotates at a rate proportional to the rate of rotation. During investigations of the effect in various solid and fluid-filled objects of various shapes, an interesting commonality was found in connection with the gyroscopic effects of the rotating object. The effect has also been discussed in connection with a rotating fluid-filled wineglass. A linear theory is developed, assuming that the rotation rate is constant and much smaller than the lowest eigenfrequency of the vibrating system. The associated physics and mathematics are easy enough for undergraduate students to understand.

[DOI: 10.1119/1.3088877]

I. INTRODUCTION

When a vibrating structure is subjected to a rotation at an angular rate Ω, the vibrating pattern rotates with respect to the structure at a rate proportional to Ω. This effect, known as "Bryan's effect," was first observed by Bryan[1] in 1890. Bryan defined the constant of proportionality for a body consisting of a ring or cylinder for various modes of vibration as

$$\mathrm{BF} = \frac{\text{Angular rate of the vibrating pattern}}{\text{Angular rate of the vibrating body}}. \tag{1}$$

The constant of proportionality BF is known as "Bryan's factor." Estimates based on Bryan's effect were used to demonstrate that the resonance of liquid surface vibrations in a wineglass[2] was predictable using a membrane model.[3]

We have been investigating Bryan's effect in various solid and fluid-filled symmetric objects that rotate at a constant rate which is much smaller than the lowest frequency of vibration of the structure. To understand Bryan's effect, investigations were conducted starting with a slowly rotating and vibrating (isotropic solid) disc and then progressing to a cylinder and a sphere. Each of these investigations yielded identical (up to constant coefficients) ordinary differential equations that can be used to explain Bryan's effect. In this paper we demonstrate how these differential equations are derived, how Bryan's factor can be calculated, and how Bryan's effect can be predicted.

In 1988, Zhuravlev and Klimov[4] investigated Bryan's effect for an isotropic, spherically symmetric body rotating in three dimensions. Among other results, they demonstrated that Bryan's effect depends on the vibration mode. Bryan's effect has numerous navigational applications.[5] Bryan's factor is used to calibrate vibrating cylindrical gyroscopes. In Ref. 5 a thin cylindrical shell was considered for both high and low rotational rates. Apart from navigational applications, the theory presented in Ref. 6 could be useful in understanding the dynamics of pulsating stars and earthquakes.

We will discuss Bryan's effect for a symmetrically distributed annular disc, where both radial and tangential vibrations are considered, and ignore axial vibrations. The theory is readily adapted to an isotropic solid cylinder (or sphere) in the form of concentric cylindrical (or spherical) bodies where some of the layers are fluids.

II. TRUE VELOCITY

Consider a body consisting of a solid disk with distributed parameters as depicted in Fig. 1. Let N be the number of concentric annular layers in the system and a_{i-1} and a_i the inner and outer radii of the ith annulus each with density ρ_i, thickness h_i, modulus of elasticity E_i, and Poisson's ratio ν_i, $i=1,\ldots,N$ [see Eqs. (A3) and (A4)]. Assume that the disk is subjected to nondecaying tangential and radial vibrations in one of its natural modes and that vibration is absent along the z-axis. In polar coordinates (with $x=r\cos\varphi$ and $y=r\sin\varphi$) consider the equilibrium position $(x,y)\equiv P(r,\varphi)$ of a vibrating particle (vibrating mass element) in the ith layer of the body, $a_{i-1}\leqslant r\leqslant a_i$. Let $\hat{\mathbf{r}}$ be the unit vector in the direction of increasing r, so that the position vector of the equilibrium point $P(r,\varphi)$ is $\mathbf{r}=r\hat{\mathbf{r}}$. Consider the orthogonal unit vector $\hat{\boldsymbol{\varphi}}=(\partial\mathbf{r}/\partial\varphi)/|\partial\mathbf{r}/\partial\varphi|$. Let $v_i\hat{\boldsymbol{\varphi}}+u_i\hat{\mathbf{r}}$ represent the displacement from the equilibrium position of the vibrating particle in the ith layer. For simplicity we suppress the subscript i if no confusion is expected. The position vector of the vibrating particle is thus

$$\mathbf{R} = (r+u)\hat{\mathbf{r}} + v\hat{\boldsymbol{\varphi}}. \tag{2}$$

Now consider an inertial coordinate system $OXYZ$ with its origin O at the center of the disc, where the X-, Y-, Z-axes initially correspond to the x-, y-, z-axes, respectively.[7] Assume that the disk rotates about the Z-axis with a small constant angular frequency Ω. Consequently, the z-axis and the Z-axis are identical, but the angle between the X-axis (which is fixed in space) and the x-axis (which is fixed with respect to the geometry of the disc) increases at a rate Ω. The angular velocity of the disk is thus

$$\boldsymbol{\Omega} = \Omega\hat{\mathbf{k}}, \tag{3}$$

where $\hat{\mathbf{k}}=\hat{\mathbf{r}}\times\hat{\boldsymbol{\varphi}}$ is the unit vector in the direction of the positive Z-axis. We assume that the angular rate of rotation Ω is substantially smaller than the lowest vibration frequency of the system. Consequently, we will neglect centrifugal effects and all other terms of $O(\Omega^2)$.

An observer in the $Oxyz$ coordinate system will measure the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$ to be constants. Hence this observer will use Eq. (2) to calculate the velocity $\mathbf{V}^*$ of the vibrating particle in the rotating framework $Oxyz$, by differentiating $\mathbf{R}$, treating $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$ as constants:

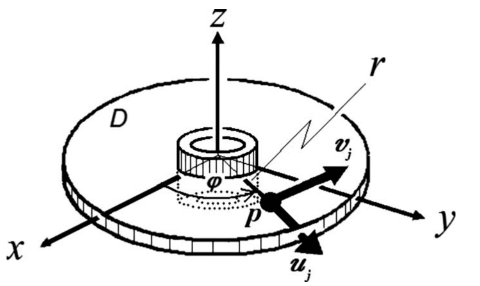

Fig. 1. Coordinate system for the annular disk consisting of various concentric annular layers of varying thickness.

$$\mathbf{V}^* = \left.\frac{d\mathbf{R}}{dt}\right|_{\hat{\mathbf{r}},\hat{\boldsymbol{\varphi}}=\text{const}} = \dot{u}\hat{\mathbf{r}} + \dot{v}\hat{\boldsymbol{\varphi}}. \tag{4}$$

An observer in the $OXYZ$ coordinate system will note that the direction of the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\varphi}}$ is continuously changing. Hence, they are not constant vectors in the $OXYZ$ frame.

In addition to the velocity $\mathbf{V}^*$, we must take into account the velocity imparted by rotation. Recall that a particle moving along a circular path of radius r and with angular rotation rate ω has a tangential speed ωr. Hence, a particle with angular velocity $\boldsymbol{\Omega}$ and position vector $\mathbf{R}$ has a velocity component given by the cross product $\boldsymbol{\Omega}\times\mathbf{R}$. Consequently, the "true velocity" of the vibrating particle as observed from within the fixed frame $OXYZ$ is

$$\mathbf{V} = \mathbf{V}^* + \boldsymbol{\Omega}\times\mathbf{R} \tag{5}$$

$$= (\dot{u} - \Omega v)\hat{\mathbf{r}} + [\dot{v} + \Omega(r+u)]\hat{\boldsymbol{\varphi}}. \tag{6}$$

Spiegel[7] provides a detailed discussion of the derivation of Eq. (5).

III. KINETIC AND POTENTIAL ENERGY

If we use Eq. (6), the kinetic energy E_k of the system of particles forming the concentric annular layers is given by

$$E_k = \frac{1}{2}\sum_{i=1}^{N}\rho_i h_i \int_0^{2\pi}\int_{a_{i-1}}^{a_i} \mathbf{V}_i\cdot\mathbf{V}_i r\,dr\,d\varphi \tag{7a}$$

$$\approx \frac{1}{2}\sum_{i=1}^{N}\rho_i h_i \int_0^{2\pi}\int_{a_{i-1}}^{a_i} [(\dot{u}_i^2 + \dot{v}_i^2) + 2\Omega(u_i\dot{v}_i - \dot{u}_i v_i) + 2\Omega\dot{v}_i r] r\,dr\,d\varphi. \tag{7b}$$

When a spring is stretched, the elastic forces involved can do work. Elastic forces are present when an "elastic" body vibrates, and so it is necessary to introduce some of the theory of elasticity to calculate the potential energy of the system of particles forming the concentric annular layers. A short discussion of elasticity is given in Appendixes A–C. According to Eq. (B4), the potential energy E_p of a system of concentric annular layers is given by

$$E_p = \frac{1}{2}\sum_{i=1}^{N} h_i \int_0^{2\pi}\int_{a_{i-1}}^{a_i} [\sigma_{r,i}\epsilon_{r,i} + \sigma_{\varphi,i}\epsilon_{\varphi,i} + \tau_{r\varphi,i}\gamma_{r\varphi,i}] r\,dr\,d\varphi, \tag{8}$$

where the symbols σ and τ stand for the tensile stress and shear stress, respectively, and ϵ and γ stand for tensile strain, and shear strain respectively. According to Eqs. (C3) and (C4), the stresses are

$$\sigma_{r,i} = \frac{E_i}{1-\nu_i^2}(\epsilon_{r,i} + \nu_i\epsilon_{\varphi,i}), \quad \sigma_{\varphi,i} = \frac{E_i}{1-\nu_i^2}(\epsilon_{\varphi,i} + \nu_i\epsilon_{r,i}), \tag{9}$$

$$\tau_{r\varphi,i} = \frac{E_i}{2(1+\nu_i)}\gamma_{r\varphi,i}. \tag{10}$$

Strains may be calculated as follows (see, for instance, Ref. 10 or 11):

$$\epsilon_{r,i} = \frac{\partial u_i}{\partial r}, \quad \epsilon_{\varphi,i} = \frac{1}{r}\left(\frac{\partial v_i}{\partial\varphi} + u_i\right), \tag{11}$$

$$\gamma_{r\varphi,i} = \frac{\partial v_i}{\partial r} + \frac{1}{r}\left(\frac{\partial u_i}{\partial\varphi} - v_i\right). \tag{12}$$

Problem 1. Substitute Eqs. (9) and (10) into Eq. (8), and then use Eqs. (11) and (12) to obtain

$$E_p = \frac{1}{2}\sum_{i=1}^{N}\frac{E_i h_i}{1-\nu_i^2}\int_0^{2\pi}\int_{a_{i-1}}^{a_i}\left\{\left[\frac{\partial u_i}{\partial r}\right]^2 + \left[\frac{1}{r}\left(\frac{\partial v_i}{\partial\varphi} + u_i\right)\right]^2 + \frac{2\nu_i}{r}\frac{\partial u_i}{\partial r}\left(\frac{\partial v_i}{\partial\varphi} + u_i\right) + \frac{1-\nu_i}{2}\left[\frac{\partial v_i}{\partial r} + \frac{1}{r}\left(\frac{\partial u_i}{\partial\varphi} - v_i\right)\right]^2\right\} r\,dr\,d\varphi. \tag{13}$$

IV. GYROSCOPIC EFFECTS IN DISTRIBUTED BODIES

Equations of motion for the vibrating particle in the ith body can be obtained by using Eqs. (9)–(12), and the equations of motion discussed by Redwood.[8] The resulting equations consist of two coupled partial differential equations involving terms such as $\partial^2 u_i/\partial t^2$, $(\partial^2 v_i/\partial t^2)$, $(\partial u_i/\partial r)$, $\partial v_i/\partial\varphi$, $\partial^2 u_i/\partial r\partial\varphi$. Solving this coupled system of partial differential equations is a nontrivial problem that involves finding, for each i, two families of eigenfunctions $U_{i,m}(r)$ and $V_{i,m}(r)$, $m=2,3,4,\ldots$. The number m is the vibration mode number or the circumferential wave number. We will not attempt to determine these eigenfunctions here, and we leave this determination for a future paper. We will assume that we can calculate these eigenfunctions and that we can (for each mode of vibration) express the displacements u_i and v_i of a vibrating particle in the ith layer of the body as follows:

$$u_i(r,\varphi,t) = U_i(r)[C(t)\cos m\varphi + S(t)\sin m\varphi], \tag{14}$$

$$v_i(r,\varphi,t) = V_i(r)[C(t)\sin m\varphi - S(t)\cos m\varphi], \tag{15}$$

$m=2,3,4,\ldots$, where the functions $C(t)$ and $S(t)$ are to be determined. Here, for simplicity, we have suppressed the mode number on the eigenfunctions, that is, $U_i(r)=U_{i,m}(r)$ and $V_i(r)=V_{i,m}(r)$. It is left as an exercise to determine the nature of the functions $C(t)$ and $S(t)$.

Problem 2. Substitute Eqs. (14) and (15) into Eqs. (7b) and (13). Simplification of these expressions involves a long algebraic calculation. The use of a computer algebra system such as Mathematica or Maple yields

$$E_k = \pi[I_0(\dot{C}^2 + \dot{S}^2) + 2\Omega I_1(\dot{C}S - C\dot{S})] \tag{16}$$

and

$$E_p = \pi I_2(C^2 + S^2), \tag{17}$$

where

$$I_0 = \frac{1}{2}\sum_{i=1}^{N} h_i\rho_i \int_{a_{i-1}}^{a_i} (U_i^2 + V_i^2) r\, dr, \tag{18}$$

$$I_1 = \sum_{i=1}^{N} h_i\rho_i \int_{a_{i-1}}^{a_i} U_i V_i r\, dr, \tag{19}$$

and

$$I_2 = \frac{1}{2}\sum_{i=1}^{N} \frac{E_i h_i}{1-\nu_i^2} \int_{a_{i-1}}^{a_i} \left\{ (U_i')^2 + 2\nu_i U_i' \frac{U_i + mV_i}{r} + \left(\frac{U_i + mV_i}{r}\right)^2 + \frac{1-\nu_i}{2}\left(V_i' - \frac{mU_i + V_i}{r}\right)^2 \right\} r\, dr. \tag{20}$$

A. Lagrange's equations

The Lagrangian follows from Eqs. (16) and (17):

$$L(C,S,\dot{C},\dot{S}) = E_k - E_p = \pi[I_0(\dot{C}^2 + \dot{S}^2) - 2\Omega I_1(C\dot{S} - \dot{C}S) - (C^2 + S^2)I_2]. \tag{21}$$

The vibration of the mth mode is governed by Lagrange's equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{C}}\right) - \frac{\partial L}{\partial C} = 0, \tag{22}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{S}}\right) - \frac{\partial L}{\partial S} = 0. \tag{23}$$

Equations (22) and (23) yield

$$\ddot{C} + 2\eta\Omega\dot{S} + \omega^2 C = 0 \tag{24}$$

and

$$\ddot{S} - 2\eta\Omega\dot{C} + \omega^2 S = 0, \tag{25}$$

respectively, where, for the mth mode of vibration

$$-1 \le \eta = \frac{I_1}{I_0} \le 1, \tag{26}$$

and ω is given by

$$\omega = \sqrt{\frac{I_2}{I_0}}. \tag{27}$$

B. Bryan's factor

We now show that ω is an eigenvalue of the vibrating system and that η in Eq. (26) is Bryan's factor BF in Eq. (1). To interpret what Eqs. (24) and (25) represent we combine the two equations by considering the complex function

$$Z = C + iS \tag{28}$$

to obtain the single equation

$$\ddot{Z} - i(2\eta\Omega)\dot{Z} + \omega^2 Z = 0. \tag{29}$$

If we write Z in polar form

$$Z(t) = r(t)e^{i\theta(t)}, \tag{30}$$

and assume that $\theta(t)$ has the linear form $\theta(t) = at$, we obtain

$$\ddot{r} + 2i(a - \eta\Omega)\dot{r} + (2\eta\Omega a - a^2 + \omega^2)r = 0. \tag{31}$$

If we choose $a = \eta\Omega$, the coefficient of $\dot{r}$ is eliminated in Eq. (31), and we obtain the differential equation of a harmonic oscillator:

$$\ddot{r} + \gamma^2 r = 0, \tag{32}$$

where

$$\gamma = \sqrt{\omega^2 + \eta^2\Omega^2} \tag{33}$$

is an eigenvalue of the vibrating system with eigenfrequency of vibration $f = \gamma/2\pi$. According to the assumption made shortly after Eq. (3), $\Omega \ll f$. Consequently,

$$\gamma \approx \omega, \tag{34}$$

and so ω is an eigenvalue of the vibrating system. Equations (24) and (25) can now be viewed in the form

$$Z(t) = r(t)e^{i\eta\Omega t}. \tag{35}$$

Equation (35) shows that Eqs. (24) and (25) represent a "vector" in the complex plane with its magnitude varying like a harmonic oscillator and its position varying at a rate proportional to the constant, small rotation rate Ω of the isotropic body. Hence, according to Eq. (1), Bryan's factor

$$\mathrm{BF} = \frac{\eta\Omega}{\Omega} = \eta. \tag{36}$$

Consequently, if a gyroscope based on Bryan's effect[12] is to be calibrated, then, without conducting lengthy experiments, Bryan's factor can be calculated from Eq. (26) once the eigenfunctions of Eqs. (14) and (15) are known.

Equations (18)–(20), (26), and (27) show that for the mth mode of vibration, Bryan's factor and the eigenfrequency of vibration depend on physical properties such as the density and geometrical properties such as thickness. The eigenfrequency also depends on elastic properties such as Young's modulus and Poisson's ratio.

Equation (35) defines a precessing wave. The rotating vibration pattern lags behind the position of the static vibration pattern if $\eta < 0$ and precedes the position of the static vibration pattern if $\eta > 0$. A calculation of η for a liquid filled wineglass[3] and $m=2$ reveals η to be negative. Hence, the rotating vibration pattern should lag behind the static vibration pattern for the wineglass.

We note that Eqs. (24) and (25) are obtained with appropriate values of I_0, I_1, and I_2 for isotropic cylindrical or spherical distributed bodies. The definite integrals I_0, I_1, and I_2 are far more complicated for a cylinder and sphere.

Problem 3. Show that to a good approximation

$$C(t) = \cos \eta\Omega t(A \cos \omega t + B \sin \omega t), \tag{37}$$

$$S(t) = \sin \eta\Omega t(A \cos \omega t + B \sin \omega t) \tag{38}$$

(where A and B are arbitrary constants) by solving Eq. (32) for $r(t)$, substituting into Eq. (35), equating real and imaginary parts, and then using Eq. (34).

Problem 4. Use the Lagrangian L as given by Eq. (21) and include viscous damping by introducing Rayleigh's dissipation function $\mathcal{F}=(c\dot{C}^2+s\dot{S}^2)/2$ into Lagrange's equations (see Ref. 9). Assume weak, isotropic, viscous damping, that is, $c=s=\pi D$, with the damping factor $\delta=D/(2I_0)$ much smaller than the lowest eigenfrequency of the vibrating system. Conclude that the introduction of light, viscous, isotropic damping into the considerations does not alter the fact that the damped vibrating pattern rotates at a rate $\eta\Omega$ in the $Oxyz$ plane, where η is given by Eq. (26). See Ref. 6 for details.

V. CONCLUSION

By using standard concepts of physics such as kinetic energy, potential energy, and Lagrange's equations, we have demonstrated how Bryan's effect for a composite disk that is rotating slowly in space can be predicted and Bryan's factor can be calculated. These considerations also demonstrated that Bryan's factor depends on properties such as the density and the thickness of the disk and that the eigenfrequency of vibration of the disk also depends on elastic properties such as Young's modulus and Poisson's ratio.

We can now better understand the operation and calibration of the hemispherical resonator gyroscope of Loper and Lynch.[12] Roughly speaking, suppose that a vibrating hemisphere is fixed to a vehicle (such as a space shuttle or submarine) moving through three-dimensional space and that a sensor inside the vehicle observes the position of a node of the fundamental vibration of the hemisphere (such vibrations can be observed in the excellent holographic interferograms of a vibrating wineglass in Ref. 13). Suppose the vehicle undergoes a slow rate of rotation Ω with respect to the space through which it is moving and that this rotation rate is too small for the human vestibular system to observe. The sensor will register that the node rotates away from its original position. From observations within the vehicle the rotation rate α of the node can be calculated and, using Bryan's factor η for the fundamental mode of vibration, the rate of rotation of the vehicle $\Omega=\alpha/\eta$ with respect to the space through which it is moving can be calculated.

ACKNOWLEDGMENTS

The authors would like to thank the Tshwane University of Technology, the National Research Foundation of South Africa (NRF Grant reference No. ICD2006071100000607), and the South African CSIR for supporting this work.

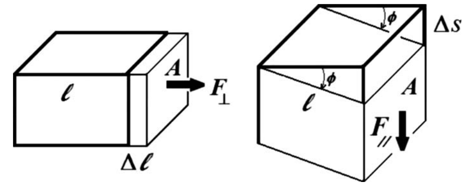

Fig. 2. Elastic blocks undergoing tensile deformation (left block) and shear deformation (right block).

APPENDIX A: ELASTIC CONSTANTS

A body will deform when stretching and/or twisting forces are applied to it. We consider as a first approximation a perfectly elastic body that returns to its original form after stretching and/or twisting forces are removed from it.

Consider a length ℓ of an elastic block (Fig. 2) with cross-sectional area A that is subjected to a stretching force $F_\perp$ (normal to the area A) causing the side length to increase from ℓ to $\ell+\Delta\ell$. The tensile stress σ of the elastic body is given by

$$\sigma = \frac{F_\perp}{A}, \tag{A1}$$

and the tensile strain ϵ is given by

$$\epsilon = \frac{\Delta\ell}{\ell}. \tag{A2}$$

Young's modulus (or the modulus of elasticity) E is given by

$$E = \frac{\text{tensile stress}}{\text{tensile strain}} = \frac{\sigma}{\epsilon}. \tag{A3}$$

If a length of elastic body is stretched from length ℓ to $\ell+\Delta\ell$, its transverse dimensions (its height or breadth) t decreases from t to $t+\Delta t$ (here $\Delta t<0$). Hence, both longitudinal strain $\Delta\ell/\ell$ and transverse strain $\Delta t/t$ are present simultaneously. (For isotropic substances transverse strain is the same for any transverse dimensions such as height, breadth, or diameter.) Poisson's ratio ν is defined as the positive dimensionless constant

$$\nu = -\left(\frac{\text{longitudinal strain}}{\text{transverse strain}}\right) = -\left(\frac{\Delta\ell/\ell}{\Delta t/t}\right). \tag{A4}$$

Suppose that an elastic block (Fig. 2) is subjected to a shearing or twisting force $F_\parallel$ (parallel to the area A) that twists the body through a small angle ϕ (in Fig. 2, $\phi \approx \Delta s/\ell$). The shear stress τ of the body is given by

$$\tau = \frac{F_\parallel}{A}, \tag{A5}$$

and the shear strain γ is given by

$$\gamma = \phi. \tag{A6}$$

The shear modulus G is given by

$$G = \frac{\text{shear stress}}{\text{shear strain}} = \frac{\tau}{\gamma}. \tag{A7}$$

It can be shown (see Ref. 10) that

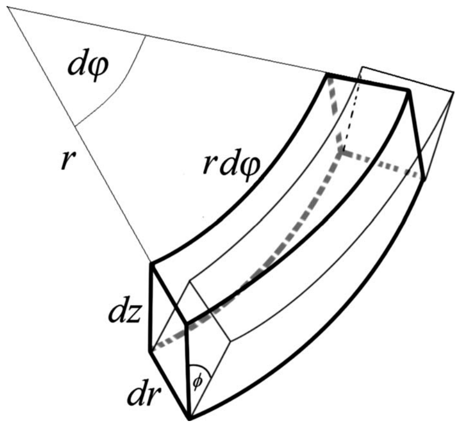

Fig. 3. Volume element $dV = rdrd\varphi dz$ in polar coordinates before deformation (thick lines) and after deformation (thin lines).

$$G = \frac{E}{2(1+\nu)}, \tag{A8}$$

and so

$$\gamma = \frac{2(1+\nu)}{E}\tau. \tag{A9}$$

Equations (A3) and (A7) are forms of Hooke's law. For instance, from Eq. (A3), we deduce the well known law for elastic elongation w (usually referred to as Hooke's law):

$$F = kw, \tag{A10}$$

with $k = AE/\ell$, $w = \Delta\ell$, and $F = F_{\perp}$.

APPENDIX B: POTENTIAL ENERGY

When a tensile extension w occurs, work has been done by the force $F = F_{\perp}$. This work is stored as potential energy. According to the tensile form of Hooke's law given by Eq. (A10), this tensile potential energy (also called tensile strain energy) is given by

$$\int_0^u F\,dw = k\int_0^u w\,dw = \frac{1}{2}(kw)[w] = \frac{1}{2}(F_{\perp}) \times [\Delta\ell] \tag{B1a}$$

$$= \frac{1}{2}(\text{tensile force}) \times [\text{tensile extension}]. \tag{B1b}$$

A similar formula holds for shear potential energy. In an elastic solid disk with distributed parameters as depicted in Fig. 1, suppose that we have an elastic volume element dV at the point P,

$$dV = drdzrd\varphi = rdrd\varphi dz, \tag{B2}$$

as depicted in Fig. 3. Here the thickness of the disk at point P is $h = \int_0^h dz$.

Tensile stresses σ_r in the radial direction and σ_φ in the tangential direction exist, but there are no tensile stresses on faces (areas) parallel to the area $rd\varphi dr$ in the $r\varphi$-plane. There are no shear stresses parallel to the areas $drdz$ or $rd\varphi dz$, but there is a shear stress $\tau_{r\varphi}$ parallel to the $r\varphi$-plane. Suppose that the volume element dV is subjected to a shear force $\tau_{r\varphi} rd\varphi dr$ which produces a shear extension $\gamma_{r\varphi} dz$. According to the shear version of Eq. (B1b), the work done by the shear force to produce this shear extension is

$$\frac{1}{2}(\text{shear force})[\text{shear extension}] = \frac{1}{2}(\tau_{r\varphi} rd\varphi dr)[\gamma_{r\varphi} dz]. \tag{B3}$$

There are two similar expressions for the work done by the radial and tangential tensile forces. We sum the three expressions to obtain the total potential (or strain) energy of the volume element:

$$dW = \frac{1}{2}(\sigma_r\epsilon_r + \sigma_\varphi\epsilon_\varphi + \tau_{r\varphi}\gamma_{r\varphi})rdrd\varphi dz. \tag{B4}$$

APPENDIX C: SUPERPOSITION

Consider the tensile and shear strains of the volume element dV of Eq. (B2). A radial (tangential) strain σ_r/E (σ_φ/E) is accompanied by a lateral contraction per unit length or a lateral strain in the tangential (radial) direction $-\nu\sigma_r/E$ ($-\nu\sigma_\varphi/E$), where ν is Poisson's ratio. Shear stresses do not cause lateral stresses. Hence, by superposition, the net strains are

$$\epsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\varphi), \quad \epsilon_\varphi = \frac{1}{E}(\sigma_\varphi - \nu\sigma_r), \tag{C1}$$

$$\gamma_{r\varphi} = \frac{1}{G}\tau_{r\varphi} = \frac{2(1+\nu)}{E}\tau_{r\varphi}. \tag{C2}$$

Problem 5. Solve Eq. (C1) simultaneously and manipulate Eq. (C2) to show that stresses are given in terms of strains as

$$\sigma_r = \frac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\varphi), \quad \sigma_\varphi = \frac{E}{1-\nu^2}(\epsilon_\varphi + \nu\epsilon_r), \tag{C3}$$

$$\tau_{r\varphi} = \frac{E}{2(1+\nu)}\gamma_{r\varphi}. \tag{C4}$$

[a]Electronic mail: joubertsv@tut.ac.za
[b]Permanent address: CSIR, Material Science and Manufacturing, Sensor Science and Technology, P. O. Box 395, Pretoria 0001, South Africa.
[1]G. H. Bryan, "On the beats in the vibrations of a revolving cylinder or bell," Proc. Cambridge Philos. Soc. Math. Phys. Sci. **7**, 101–114 (1890).
[2]R. E. Apfel, " 'Whispering waves' in a wineglass," Am. J. Phys. **53**(11), 1070–1073 (1985).
[3]S. V. Joubert, T. H. Fay, and E. L. Voges, "A storm in a wineglass," Am. J. Phys. **75**(7), 647–651 (2007).
[4]V. Zhuravlev and D. Klimov, *Applied Methods in the Theory of Oscillations* (Nauka, Moscow, 1988) (in Russian).
[5]P. W. Loveday and C. A. Rogers, "Free vibrations of elastically supported thin cylinders including gyroscopic effects," J. Sound Vib. **217**(3), 547–562 (1998).
[6]M. Y. Shatalov, S. V. Joubert, C. E. Coetzee, and I. A. Fedotov, "Free vibration of rotating hollow spheres containing acoustic media," J. Sound Vib., to be published. AQ: #1
[7]M. R. Spiegel, *Theoretical Mechanics* (McGraw-Hill, New York, 1967), pp. 148, 284.
[8]M. Redwood, *Mechanical Waveguides* (Pergamon, Oxford, 1960), pp. 9, 268.
[9]H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics*, 3rd ed. (Addison-Wesley, Reading, MA, 2001), pp. 21, 23.

[10] S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity* (McGraw-Hill, Auckland, 1970), pp. 8–11, 75–77, 244–245.
[11] P. P. Benham and R. J. Crawford, *Mechanics of Engineering Materials* (Longman Scientific, Harlow, 1991), p. 381.
[12] E. J. Loper, Jr. and D. D. Lynch, "Hemispherical resonator gyroscope," U.S. Patent No. 4,951,508.
[13] T. D. Rossing, "Acoustics of the glass harmonica," J. Acoust. Soc. Am. **95** (2), 1106–1111 (1994).

AUTHOR QUERIES — 006905AJP

#1 Au: Please update Ref. 6, if possible.