General Problems of Dynamics and Control of Vibratory Gyroscopes

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A general model of operation of vibratory gyroscopes, which is applicable to a broad class of instruments, including cylindrical, disc and micro-machined gyros, is formulated on the basis of analysis of dynamics and control of a hemispherical resonator gyroscope. The foundations of feedback control in the gyroscopes are considered and classification of the main operational regimes is given in terms of the integral manifolds and new classes of nonlinear parametric excitation forces are added. Qualitative effects of different classes of forces are analysed and conclusions about the structure of nonlinear control forces are proposed. Specific imperfections of the vibratory gyroscopes, operating in the whole regime, force-to-rebalance, travelling wave and combined regimes, are considered. The cumulative effects of errors are represented as superposition of the effects of particular classes. A closed set of integral manifolds is formulated, in terms of which the dynamics of a vibratory gyroscope is considered.

1. Kinematics of the Gyroscope

Location of an element E on a spherical surface can be characterized by two angles φ and θ; the rule of coordinates transformation is as follows (*See* Fig.1):

$$O\xi\eta\zeta \xrightarrow[(O\zeta)]{\varphi} Ox_1y_1z_1 \xrightarrow[(Oy_1)]{\theta} Oxyz \quad , \tag{1}$$

where $O\xi\eta\zeta$ is a reference coordinate system, connected with the resonator;

$Ox_1y_1z_1$ is a coordinate system, connected with the resonator and turned with respect to $\tilde{O}$ at the angle φ;

$Oxyz$ is a coordinate system, connected with the resonator and turned with respect to Oy_1 at the angle θ;

φ is a polar or azimuthal angle of the element dS;

θ is its latitudinal angle (Fig.1).

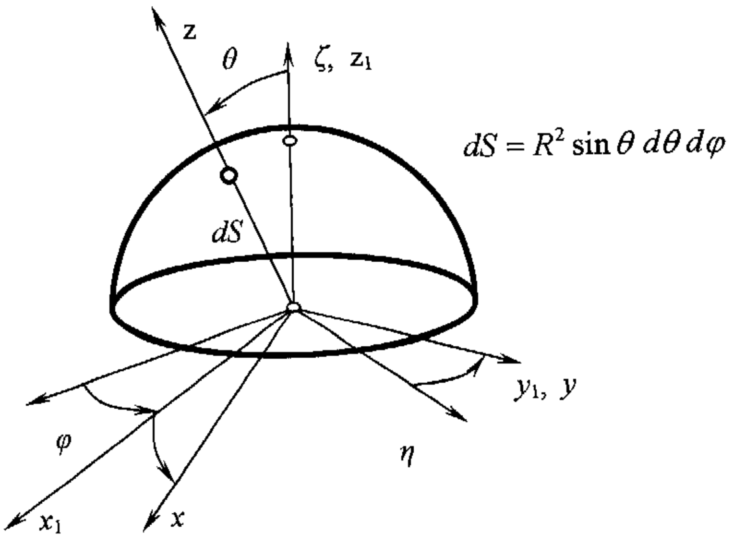

Fig.1.

Absolute linear velocity:

$$\vec{V}=\vec{\tilde{V}}+\vec{\dot{r}}+\vec{\Omega}\times\vec{r}\ , \tag{2}$$

where

$$\vec{\Omega}=\begin{bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{bmatrix}=\begin{bmatrix}\left(\Omega_\xi\cos\varphi+\Omega_\eta\sin\varphi\right)\cos\theta-\Omega_\zeta\sin\theta\\ -\Omega_\xi\sin\varphi+\Omega_\eta\cos\varphi\\ \left(\Omega_\xi\cos\varphi+\Omega_\eta\sin\varphi\right)\sin\theta+\Omega_\zeta\cos\theta\end{bmatrix} \tag{3}$$

and

$$\vec{\tilde{V}}=\begin{bmatrix}\tilde{V}_x\\ \tilde{V}_y\\ \tilde{V}_z\end{bmatrix}=\begin{bmatrix}\left(\tilde{V}_\xi\cos\varphi+\tilde{V}_\eta\sin\varphi\right)\cos\theta-\tilde{V}_\zeta\sin\theta\\ -\tilde{V}_\xi\sin\varphi+\tilde{V}_\eta\cos\varphi\\ \left(\tilde{V}_\xi\cos\varphi+\tilde{V}_\eta\sin\varphi\right)\sin\theta+\tilde{V}_\zeta\cos\theta\end{bmatrix}. \tag{4}$$

We suppose that $\frac{\Omega_{\xi,\eta,\zeta}}{\omega}<<1$ and $\frac{\tilde{V}_{\xi,\eta,\zeta}}{\omega}<<1$, where ω is a natural frequency of an operational mode. For this case, we neglect the centrifugal forces, and, hence,

$$\begin{aligned}V^2&=\left(\dot{u}^2+\dot{v}^2+\dot{w}^2\right)+2\Omega_x\left[\dot{w}v-\dot{v}\left(R+w\right)\right]+2\Omega_y\left[\dot{u}\left(R+w\right)-\dot{w}u\right]\\&+2\Omega_z\left[\dot{v}u-\dot{u}v\right]+2\left[\dot{u}\tilde{V}_x+\dot{v}\tilde{V}_y+\dot{w}\tilde{V}_z\right]\ .\end{aligned} \tag{5}$$

We also assume that n-th mode could be represented as follows:

$$\begin{aligned}u_n(\varphi,\theta,t)&=X_n(\theta)\left[a_n(t)\cos n\varphi+b_n(t)\sin n\varphi\right]\ ,\\ v_n(\varphi,\theta,t)&=Y_n(\theta)\left[-a_n(t)\sin n\varphi+b_n(t)\cos n\varphi\right]\ ,\\ w_n(\varphi,\theta,t)&=Z_n(\theta)\left[a_n(t)\cos n\varphi+b_n(t)\sin n\varphi\right]\ ;\end{aligned} \tag{6}$$

where

$$X_n(\theta)=-Y_n(\theta)=-\sin\theta\tan^n\left(\frac{\theta}{2}\right)\ ,\quad Z_n(\theta)=\left(n+\cos\theta\right)\tan^n\left(\frac{\theta}{2}\right)\ . \tag{7}$$

are the Rayleigh inextensional solutions (pseudo-bending).

2. Dynamics of the Gyroscope - 1: Lagrangian of the System

Let us consider a linear model of a particular vibrational mode of a vibratory gyroscope in terms of Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{a}_n}\right)-\frac{\partial L}{\partial a_n}=-\frac{\partial D}{\partial\dot{a}_n}+Q_a,\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{b}_n}\right)-\frac{\partial L}{\partial b_n}=-\frac{\partial D}{\partial\dot{b}_n}+Q_b\ ; \tag{8}$$

where $a=a_n(t)$, $b=b_n(t)$, $L=T-P$ is a Lagrangian of n-th vibrational mode, T is a kinetic energy, P is a strain energy, D is a Rayleigh damping function, $Q_{a,b}$ is a generalized forces, including control, noise, *etc.*

Kinetic energy of the system is as follows:

$$T=T^{(\mathrm{hem})}+T^{(\mathrm{tines})}\ , \tag{9}$$

where $T^{(\text{hem})}$ is a kinetic energy of the hemispherical part of the resonator, $T^{(\text{tines})}$ is a kinetic energy of the tines assembly:

$$T^{(\text{hem})} = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi/2} \rho(\varphi,\theta)R^2(\varphi,\theta)h(\varphi,\theta)V^2(\varphi,\theta,t)\sin\theta \, d\theta \, d\varphi \quad ,$$
$$T^{(\text{tines})} = \frac{1}{2}\sum_{i=1}^{N_t} m_i V^2\left(\varphi_i = \frac{2\pi}{N_t}i, \theta = \frac{\pi}{2}, t\right) \quad ; \tag{10}$$

where $\rho(\varphi,\theta)$ is a mass density of the resonator, $R(\varphi,\theta)$ is a radius of mid surface of the resonator, $h(\varphi,\theta)$ is a thickness of the shell, $V(\varphi,\theta,t)$ is a linear velocity, N_t is a number of tines.

Strain energy of the system is:

$$P = P^{(\text{hem})} + P^{(\text{exc-corr})} + P^{(\text{sens})} + P^{(\text{ring})} \quad , \tag{11}$$

where $P^{(\text{hem})}$ is an energy of the resonator's hemispherical part, $P^{(\text{exc-corr})}$ is an electric field energy in the gaps between the resonator and exciting-forcing electrodes, $P^{(\text{sens})}$ is an electric field energy in the gaps between the resonator and pick-off electrodes, $P^{(\text{ring})}$ is an electric field energy in the gap between the resonator and ring forcer electrode.

According to the Novozhilov's theory of thin shells:

$$\begin{aligned} P^{(\text{hem})} = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi/2} \frac{E(\varphi,\theta)R^2(\varphi,\theta)h(\varphi,\theta)}{1-\eta^2(\varphi,\theta)} \Bigg\{ \Big[\left(\varepsilon_1(\varphi,\theta,t) + \varepsilon_2(\varphi,\theta,t)\right)^2 - \\ 2\left(1-\eta(\varphi,\theta)\right)\left(\varepsilon_1(\varphi,\theta,t)\varepsilon_2(\varphi,\theta,t) - \frac{\omega^2(\varphi,\theta,t)}{4}\right)\Big] + \frac{h^2(\varphi,\theta)}{12}\Big[\left(\kappa_1(\varphi,\theta,t)+\kappa_2(\varphi,\theta,t)\right)^2 \\ -\left(1-\eta(\varphi,\theta)\right)\left(\kappa_1(\varphi,\theta,t)\kappa_2(\varphi,\theta,t) - \tau^2(\varphi,\theta,t)\right)\Big]\Bigg\}\sin\theta \, d\theta \, d\varphi \quad , \end{aligned} \tag{12}$$

where

$$\begin{aligned} &\varepsilon_1 = \frac{1}{A_1}\frac{\partial u}{\partial\theta} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\varphi}v + \frac{w}{R_1}; \quad \varepsilon_2 = \frac{1}{A_1}\frac{\partial v}{\partial\varphi} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\theta}u + \frac{w}{R_2}; \quad \omega = \frac{A_1}{A_2}\frac{\partial}{\partial\varphi}\left(\frac{u}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial\varphi}\left(\frac{v}{A_2}\right); \\ &\kappa_1 = -\frac{1}{A_1}\frac{\partial}{\partial\theta}\left(\frac{1}{A_1}\frac{\partial w}{\partial\theta} - \frac{u}{R_1}\right) - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\varphi}\left(\frac{1}{A_2}\frac{\partial w}{\partial\varphi} - \frac{v}{R_2}\right); \quad \kappa_2 = -\frac{1}{A_2}\frac{\partial}{\partial\varphi}\left(\frac{1}{A_2}\frac{\partial w}{\partial\varphi} - \frac{v}{R_2}\right) - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\varphi}\left(\frac{1}{A_1}\frac{\partial w}{\partial\theta} - \frac{u}{A_1}\right); \\ &\tau = \frac{1}{A_1A_2}\left(\frac{\partial^2 w}{\partial\varphi\partial\theta} - \frac{1}{A_1}\frac{\partial A_1}{\partial\varphi}\frac{\partial w}{\partial\theta} - \frac{1}{A_2}\frac{\partial A_2}{\partial\theta}\frac{\partial w}{\partial\varphi}\right) + \frac{1}{R_1}\left(\frac{1}{A_2}\frac{\partial u}{\partial\varphi} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\varphi}u\right) + \frac{1}{R_2}\left(\frac{1}{A_1}\frac{\partial v}{\partial\theta} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\theta}v\right); \end{aligned} \tag{13}$$

and $A_1 = R(\varphi,\theta)$, $A_2 = R(\varphi,\theta)\sin\theta$, $R_1 = R_2 = R(\varphi,\theta)$.

Other components of the strain energy are as follows:

$$P^{(\text{exc-corr})} = -\frac{\varepsilon_0}{2}\sum_{i=1}^{N_t} U_{ci}^2(t)\int_{\varphi_{1i}}^{\varphi_{2i}}\int_{\theta_1(\varphi)}^{\theta_2(\varphi)} \frac{\left[R(\varphi,\theta) + \left(h(\varphi,\theta)+\Delta_{ci}(\varphi,\theta)\right)/2\right]^2}{\Delta_{ci}(\varphi,\theta) - w(\varphi,\theta,t)}\sin\theta \, d\theta \, d\varphi \; ,$$

$$P^{(\text{sens})} = -\frac{\varepsilon_0}{2}\sum_{i=1}^{N_s} U_{si}^2(t)\int_{\varphi_{3i}}^{\varphi_{4i}}\int_{\theta_3(\varphi)}^{\theta_4(\varphi)} \frac{\left[R(\varphi,\theta) - {\left(h(\varphi,\theta)+\Delta_{si}(\varphi,\theta)\right)}\big/_2\right]^2}{\Delta_{si}(\varphi,\theta)+w(\varphi,\theta,t)}\sin\theta\, d\theta\, d\varphi \ ,$$

$$P^{(\text{ring})} = -\frac{\varepsilon_0}{2}U_r^2(t)\int_0^{2\pi}\int_{\theta_5(\varphi)}^{\theta_6(\varphi)} \frac{\left[R(\varphi,\theta) + {\left(h(\varphi,\theta)+\Delta_r(\varphi,\theta)\right)}\big/_2\right]^2}{\Delta_r(\varphi,\theta)-w(\varphi,\theta,t)}\sin\theta\, d\theta\, d\varphi \ ; \qquad (14)$$

where N_s is a number of pick-off electrodes, $\Delta_{ci}(\varphi,\theta)$, $\Delta_{si}(\varphi,\theta)$, $\Delta_r(\varphi,\theta)$ are initial gaps between the resonator and exciting-correcting, pick-off, and ring electrodes correspondingly, $U_{ci}(t), U_{si}(t), U_r(t)$ is a difference of potentials between the resonator and exciting-correcting, pick-off, and ring electrodes correspondingly.

Rayleigh damping function is introduced phenomenologically as:

$$D = D\left(\dot a_n, \dot b_n\right) = \int_0^{2\pi}\int_0^{\pi/2} d(\varphi,\theta)\left[\frac{\dot a_n^2+\dot b_n^2}{2}+\frac{\dot a_n^2-\dot b_n^2}{2}\cos 2n\varphi + \dot a_n \dot b_n \sin 2n\varphi\right] d\theta\, d\varphi \ ,(15)$$

where $d(\varphi,\theta)$ is a coefficient of viscous damping.

3. Dynamics of the Gyroscope - 2: Perfect Gyro

Dynamics of perfect gyro is described by the following system of equations:

$$\begin{aligned} &\frac{d^2a}{dt^2} - 2\beta\,\Omega(t)\frac{db}{dt} - \beta\frac{d\Omega(t)}{dt}b + \omega^2(t)a = 0 \ , \\ &\frac{d^2b}{dt^2} + 2\beta\,\Omega(t)\frac{da}{dt} + \beta\frac{d\Omega(t)}{dt}a + \omega^2(t)b = 0 \ ; \end{aligned} \qquad (16)$$

where $\Omega(t)$ is a projection of angular rate of inertial rotation of the gyro on its input axis (input angular rate), β is a *Bryan-Loper-Lynch factor,* describing the *Bryan-Loper-Lynch effect, i.e.,* a rotation of a vibrating pattern with regards to the resonator at presence of an inertial rotation of the gyro with variable input angular rate.

System (16) could be rewritten as

$$\frac{d^2\vec z}{dt^2} + \omega^2\vec z = \vec F\left(\vec z, \frac{d\vec z}{dt}, t\right), \qquad (17)$$

where $\vec z = (a,b)^{\mathrm T}$, $\vec F = (F_a, F_b)^{\mathrm T} = \sum_{j=1}^{3}\vec F_j\left(\vec z, \dot{\vec z}, t\right)$, with particular forces $\vec F_j$ from the following three classes:

<u>1 Class:</u> *Gyroscopic forces:*

$$\vec F_1\left(\frac{d\vec z}{dt}, t\right) = 2\gamma(t)\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}\frac{d\vec z}{dt} \qquad (18)$$

where $\gamma(t) = \beta\,\Omega(t)$.

2 Class: *Circular forces:*

$$\bar{F}_2(\bar{z},t)=n(t)\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}\bar{z}\ , \tag{19}$$

where $n(t)=\beta\dot{\Omega}(t)$.

3 Class: *Spherical Potential forces:*

$$\bar{F}_3(\bar{z},t)=k(t)\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}\bar{z}\ , \tag{20}$$

where $k(t)=\omega(t)$.

Classification of forces was first introduced by Thompson & Taight. It was shown by academicians V. Zhuravlev and D. Klimov that this approach is very convenient for the analysis of the vibratory gyroscopes dynamics.

4. Dynamics of the Gyroscope - 3: Influence of Imperfections

There are several classes of forces, which are responsible for particular imperfections.

4 Class: *Spherical Viscous Damping forces:*

$$\bar{F}_4\left(\frac{d\bar{z}}{dt},t\right)=-2\delta\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix}\frac{d\bar{z}}{dt}\ , \tag{21}$$

where δ is a viscous damping coefficient. These forces describe homogeneous amplitude decaying effects.

5 Class: *Hyperbolic Viscous Damping forces:*

$$\bar{F}_5\left(\frac{d\bar{z}}{dt},t\right)=-2\delta\begin{Vmatrix}\varepsilon_{2nC}^{(\delta)} & \varepsilon_{2nS}^{(\delta)}\\ \varepsilon_{2nS}^{(\delta)} & -\varepsilon_{2nC}^{(\delta)}\end{Vmatrix}\frac{d\bar{z}}{dt}\ , \tag{21}$$

where δ is a viscous damping coefficient. These forces describe inhomogeneous amplitude decaying effects and depends on cosine $(\varepsilon_{2nC}^{(\delta)})$ and sine $(\varepsilon_{2nS}^{(\delta)})$ $2n$-th harmonics of the viscous damping coefficient $d(\varphi,\theta)$ (*see* (15)).

6 Class: *Hyperbolic Potential forces:*

$$\bar{F}_6(\bar{z},t)=-\omega^2\begin{Vmatrix}\varepsilon_{2nC}^{(m-s)} & \varepsilon_{2nS}^{(m-s)}\\ \varepsilon_{2nS}^{(m-s)} & -\varepsilon_{2nC}^{(m-s)}\end{Vmatrix}\bar{z}\ . \tag{22}$$

These forces describe influence of $2n$-th cosine $(\varepsilon_{2nC}^{(m-s)})$ and sine $(\varepsilon_{2nS}^{(m-s)})$ mass-stiffness imperfections on the gyro dynamics. They could also originate from residual prestress in the structure, as well as from the radius of mid surface and thickness variations of the resonator.

7 Class: *Positional forces:*

$$\bar{F}_7(t)=\begin{Vmatrix}F_{7a}(t)\\ F_{7b}(t)\end{Vmatrix}\ . \tag{23}$$

These forces originate from n-th harmonics of mass density, radius and thickness imperfections of the resonator subjected to angular vibrations and projection of linear vibrations on the input axis of the gyroscope. $(n-1)$-th and $(n+1)$-th harmonics of mass density could also stipulate these forces in the resonator subjected to angular vibrations and projections of linear vibrations on the plane orthogonal to the input axis. First harmonic of mass imperfection generates these forces if the resonator the inertial angular rate of rotation does not coincide with the input axis of the gyroscope. Time dependent control forces could also generate the positional forces. In all these cases the excitation frequency is close to the resonance frequency of the n-th vibrational mode.

<u>8 Class:</u> *Spherical forces of Parametric Excitation:*

$$\vec{F}_8(\vec{z},t) = p(t)\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}\vec{z} \ . \tag{24}$$

These forces are used for excitation of a vibrating pattern in the run regime of the gyroscope.

<u>9 Class:</u> *Hyperbolic forces of Parametric Excitation:*

$$\vec{F}_9(\vec{z},t) = q(t)\begin{Vmatrix} \varepsilon_{2nC}^{(p)} & \varepsilon_{2nS}^{(p)} \\ \varepsilon_{2nS}^{(p)} & -\varepsilon_{2nC}^{(p)} \end{Vmatrix}\vec{z} \ . \tag{25}$$

These forces originate from $2n$-th harmonic of the gap between the resonator and ring forcer electrode.

All the above described classes represent the linear forces. They are naturally generated in the process of operating of the gyroscope. Now we describe some additional nonlinear forces, which are mainly generated in the process of autonomous control of the gyro as well as due to presence of nonlinear forces.

<u>10 Class:</u> *Autonomous Nonlinear Positional forces:*

$$\vec{F}_{10}(\vec{z}) = k_{10}|\vec{z}|^2\begin{Vmatrix} \cos\alpha_{10} & \sin\alpha_{10} \\ -\sin\alpha_{10} & \cos\alpha_{10} \end{Vmatrix}\vec{z} \ , \tag{26}$$

where $|\vec{z}|^2 = a^2 + b^2$.

<u>11 Class:</u> *Autonomous Nonlinear Velocity forces:*

$$\vec{F}_{11}\left(\frac{d\vec{z}}{dt}\right) = k_{11}\left|\frac{d\vec{z}}{dt}\right|^2\begin{Vmatrix} \cos\alpha_{11} & \sin\alpha_{11} \\ -\sin\alpha_{11} & \cos\alpha_{11} \end{Vmatrix}\frac{d\vec{z}}{dt} \ , \tag{27}$$

where $|\dot{\vec{z}}|^2 = \dot{a}^2 + \dot{b}^2$.

<u>12 Class:</u> *Autonomous Nonlinear Combined forces (1):*

$$\vec{F}_{12}\left(\frac{d\vec{z}}{dt},\vec{z}\right) = k_{12}\left|\frac{d\vec{z}}{dt}\right|^2\begin{Vmatrix} \cos\alpha_{12} & \sin\alpha_{12} \\ -\sin\alpha_{12} & \cos\alpha_{12} \end{Vmatrix}\vec{z} \ . \tag{28}$$

<u>13 Class:</u> *Autonomous Nonlinear Combined forces (2):*

$$\vec{F}_{13}(\dot{\vec{z}},\vec{z}) = k_{13}|\vec{z}|^2\begin{Vmatrix} \cos\alpha_{13} & \sin\alpha_{13} \\ -\sin\alpha_{13} & \cos\alpha_{13} \end{Vmatrix}\frac{d\vec{z}}{dt} \ . \tag{29}$$

14 Class: ***Autonomous Nonlinear Combined forces (3):***

$$\vec{F}_{14}\left(\frac{d\bar{z}}{dt},\bar{z}\right)=k_{14}\left(\bar{z}\cdot\frac{d\bar{z}}{dt}\right)\begin{Vmatrix}\cos\alpha_{14} & \sin\alpha_{14}\\ -\sin\alpha_{14} & \cos\alpha_{14}\end{Vmatrix}\bar{z} \ , \tag{30}$$

where $\left(\bar{z}\cdot\frac{d\bar{z}}{dt}\right)=a\dot{a}+b\dot{b}$ is a scalar (dot) product of vectors $\bar{z}$ and $\dot{\bar{z}}$.

15 Class: ***Autonomous Nonlinear Combined forces (4):***

$$\vec{F}_{15}\left(\frac{d\bar{z}}{dt},\bar{z}\right)=k_{15}\left(\bar{z}\cdot\frac{d\bar{z}}{dt}\right)\begin{Vmatrix}\cos\alpha_{15} & \sin\alpha_{15}\\ -\sin\alpha_{15} & \cos\alpha_{15}\end{Vmatrix}\frac{d\bar{z}}{dt} \ . \tag{31}$$

16 Class: ***Autonomous Nonlinear Combined forces (5):***

$$\vec{F}_{16}\left(\frac{d\bar{z}}{dt},\bar{z}\right)=k_{16}\left\|\left[\bar{z}\times\frac{d\bar{z}}{dt}\right]\right\|\begin{Vmatrix}\cos\alpha_{16} & \sin\alpha_{16}\\ -\sin\alpha_{16} & \cos\alpha_{16}\end{Vmatrix}\bar{z} \ , \tag{32}$$

where $\left\|\left[\bar{z}\times\frac{d\bar{z}}{dt}\right]\right\|=a\dot{b}-\dot{a}b$ is a modulus of the vector (cross) product of vectors $\bar{z}$ and $\dot{\bar{z}}$.

17 Class: ***Autonomous Nonlinear Combined forces (6):***

$$\vec{F}_{17}\left(\frac{d\bar{z}}{dt},\bar{z}\right)=k_{17}\left\|\left[\bar{z}\times\frac{d\bar{z}}{dt}\right]\right\|\begin{Vmatrix}\cos\alpha_{17} & \sin\alpha_{17}\\ -\sin\alpha_{17} & \cos\alpha_{17}\end{Vmatrix}\frac{d\bar{z}}{dt} \ . \tag{33}$$

18 Class: ***Autonomous Nonlinear Nodal Quadrature Positional forces:***

$$\vec{F}_{18}\left(\frac{d\bar{z}}{dt},\bar{z}\right)=k_{18}\left\|\left[\bar{z}\times\frac{d\bar{z}}{dt}\right]\right\|\begin{Vmatrix}\cos\alpha_{18} & \sin\alpha_{18}\\ \sin\alpha_{18} & -\cos\alpha_{18}\end{Vmatrix}\bar{z} \ . \tag{34}$$

19 Class: ***Non-Autonomous Nonlinear Amplitude forces:***

$$\vec{F}_{19}\left(\frac{d\bar{z}}{dt},\bar{z},t\right)=k_{19}\left(|\bar{z}|^2+\omega^{-2}\left|\frac{d\bar{z}}{dt}\right|^2\right)\begin{Vmatrix}\cos\alpha_{19} & \sin\alpha_{19}\\ -\sin\alpha_{19} & \cos\alpha_{19}\end{Vmatrix}\bar{z}\cos\left(2\omega t+\psi_{19}\right) \ . \tag{35}$$

20 Class: ***Non-Autonomous Nonlinear Sine Rotational forces:***

$$\vec{F}_{20}\left(\frac{d\bar{z}}{dt},\bar{z},t\right)=\frac{k_{20}}{2}\left[\operatorname{Im}\left(Z^2\right)+\omega^{-2}\operatorname{Im}\left(\left(\frac{d\bar{z}}{dt}\right)^2\right)\right]\begin{Vmatrix}\cos\alpha_{20} & \sin\alpha_{20}\\ \sin\alpha_{20} & -\cos\alpha_{20}\end{Vmatrix}\bar{z}\cos\left(2\omega t+\psi_{20}\right) \ , \tag{36}$$

where $Z=a+ib$ is a complex number, hence $Z^2=\left(a^2-b^2\right)+i\left(2ab\right)$.

21 Class: ***Non-Autonomous Nonlinear Cosine Rotational forces:***

$$\vec{F}_{21}\left(\frac{d\bar{z}}{dt},\bar{z},t\right)=k_{21}\left[\operatorname{Re}\left(Z^2\right)+\omega^{-2}\operatorname{Re}\left(\left(\frac{d\bar{z}}{dt}\right)^2\right)\right]\begin{Vmatrix}\cos\alpha_{21} & \sin\alpha_{21}\\ \sin\alpha_{21} & -\cos\alpha_{21}\end{Vmatrix}\bar{z}\cos\left(2\omega t+\psi_{21}\right). \tag{37}$$

These forces are efficient at the stage of control laws synthesis for stabilization of particular operational regimes.

5. Integral Manifolds and their Normal Vectors

Analysis of vibratory gyroscope's dynamics on the basis of integral manifolds was proposed by V. Zhuravlev. In the present chapter we describe the main manifolds, their normal vectors, and introduce a complete system of manifolds.

Solution of corresponding homogeneous system to system (17) is as follows:

$$a(t) = z_1(t)\cos t + z_2(t)\sin t, \qquad b(t) = z_3(t)\cos t + z_4(t)\sin t \tag{38}$$

(we assume that $\omega = 1$). Hence, equation of ellipse in a, b are variables is

$$a^2\left(z_3^2 + z_4^2\right) + b^2\left(z_1^2 + z_2^2\right) - 2ab\left(z_1 z_3 + z_2 z_4\right) = \left(z_1 z_4 - z_2 z_3\right)^2. \tag{39}$$

See Fig.2.

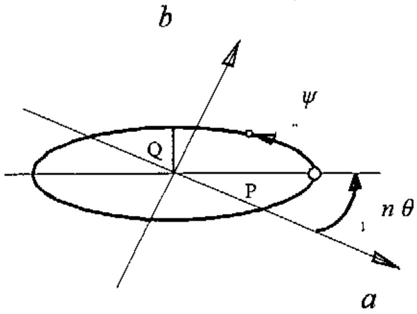

Fig.2.

It follows from (39) that if $z_1 z_4 - z_2 z_3 = 0$ the ellipse becomes the sector of straight line $b = \dfrac{z_1 z_3 + z_2 z_4}{z_1^2 + z_2^2} a$ (*pure standing wave regime*); if $z_1 z_3 + z_2 z_4 = 0$ and $z_1 z_4 - z_2 z_3 \neq 0$ the main axes of the ellipse coincide with Oa, Ob-axes (*force-to-rebalance regime*); if $z_1 z_3 + z_2 z_4 = 0$ and $z_1^2 + z_2^2 = z_3^2 + z_4^2$ the ellipse degenerates into the circle (*pure travelling wave regime*) $a^2 + b^2 = \dfrac{\left(z_1 z_4 - z_2 z_3\right)^2}{z_1^2 + z_2^2}$.

Hence, the main operational regimes could be characterized in terms of special manifolds.

Let us consider the *Friedland-Hutton orthogonal representation* of a vibrating pattern:

$$\begin{aligned} w_n(\varphi,\theta,\psi,t) &= P\cos n(\varphi-\theta)\cos(t-\psi) + Q\sin n(\varphi-\theta)\sin(t-\psi) = \\ &= a(\theta,\psi,t)\cos n\varphi + b(\theta,\psi,t)\sin n\varphi = \\ &= \left[z_1(\theta,\psi)\cos t + z_2(\theta,\psi)\sin t\right]\cos n\varphi + \left[z_3(\theta,\psi)\cos t + z_4(\theta,\psi)\sin t\right]\sin n\varphi \;, \end{aligned} \tag{40}$$

where P, Q are amplitude of principal and quadrature standing waves, θ, ψ are their orientation and phase correspondingly. They could be characterized through the manifolds:

$$R^2 = P^2 + Q^2 = z_1^2 + z_2^2 + z_3^2 + z_4^2, \qquad N = PQ = z_1 z_4 - z_2 z_3 \;, \tag{41}$$

$$\tan 2n\theta = \frac{\sin 2n\theta}{\cos 2n\theta} = \frac{2(z_1 z_3 + z_2 z_4)}{z_1^2 + z_2^2 - z_3^2 - z_4^2}, \qquad \tan 2\psi = \frac{\sin 2\psi}{\cos 2\psi} = \frac{2(z_1 z_2 + z_3 z_4)}{z_1^2 - z_2^2 + z_3^2 - z_4^2} .$$

Hence, for characterization of these values, it is enough to have six manifolds:

$$S = \frac{1}{2}(z_1^2 + z_2^2 + z_3^2 + z_4^2 - 1), \qquad N = z_1 z_4 - z_2 z_3 ,$$

$$AC = \frac{1}{2}(z_1^2 - z_2^2 + z_3^2 - z_4^2 - 1), \qquad AS = z_1 z_2 + z_3 z_4 ,$$

$$RC = \frac{1}{2}(z_1^2 + z_2^2 - z_3^2 - z_4^2 - 1), \qquad RS = z_1 z_3 + z_2 z_4 . \tag{43}$$

Of course, these manifolds are not independent:

$$4(N^2 + RS^2) + (2RC + 1)^2 = 4(N^2 + AS^2) + (2AC + 1)^2 = (2S + 1)^2 . \tag{44}$$

A complete set of integral manifolds, which is enough for characterization of the main operational regimes, is given in the table.

In this table, the normal to integral manifolds vectors are introduced as follows:

$$\vec{e}_S = \text{grad}\, S = \left[\frac{\partial S}{\partial z_1}, \frac{\partial S}{\partial z_2}, \frac{\partial S}{\partial z_3}, \frac{\partial S}{\partial z_4}\right]^{\mathrm{T}} = [z_1, z_2, z_3, z_4]^{\mathrm{T}},$$

$$\vec{e}_N = \text{grad}\, N = \left[\frac{\partial N}{\partial z_1}, \frac{\partial N}{\partial z_2}, \frac{\partial N}{\partial z_3}, \frac{\partial N}{\partial z_4}\right]^{\mathrm{T}} = [z_4, -z_3, -z_2, z_1]^{\mathrm{T}}, \quad etc. \tag{45}$$

Completeness of the manifold's set means that the scalar products of two arbitrary normal vectors results in a new manifold, which is linearly dependent on the integral manifolds given in the table (*Zhuravlev's manifolds*).

Table.

Number	**Integral Manifold**	**Normal Vector**
1.	$S = \frac{1}{2}(z_1^2 + z_2^2 + z_3^2 + z_4^2 - 1)$	$\vec{e}_S = [z_1, z_2, z_3, z_4]^{\mathrm{T}}$
2.	$N = z_1 z_4 - z_2 z_3$	$\vec{e}_N = [z_4, -z_3, -z_2, z_1]^{\mathrm{T}}$
3.	$A = AS = z_1 z_2 + z_3 z_4$	$\vec{e}_{AS} = [z_2, z_1, z_4, z_3]^{\mathrm{T}}$
4.	$AC = \frac{1}{2}(z_1^2 - z_2^2 + z_3^2 - z_4^2 - 1)$	$\vec{e}_{AC} = [z_1, -z_2, z_3, -z_4]^{\mathrm{T}}$
5.	$R = RS = z_1 z_3 + z_2 z_4$	$\vec{e}_{RS} = [z_3, z_4, z_1, z_2]^{\mathrm{T}}$
6.	$RC = \frac{1}{2}(z_1^2 + z_2^2 - z_3^2 - z_4^2 - 1)$	$\vec{e}_{RC} = [z_1, z_2, -z_3, -z_4]^{\mathrm{T}}$
7.	$T1 = z_1 z_3 - z_2 z_4$	$\vec{e}_{T1} = [z_3, -z_4, z_1, -z_2]^{\mathrm{T}}$

8. $T2=\frac{1}{2}\left(z_1^2-z_2^2-z_3^2+z_4^2-1\right)$ $\quad \vec{e}_{T2}=\left[z_1,-z_2,-z_3,z_4\right]^{\mathrm{T}}$

9. $M1=z_1 z_4+z_2 z_3$ $\quad \vec{e}_{M1}=\left[z_4,z_3,z_2,z_1\right]^{\mathrm{T}}$

10. $M2=z_1 z_2-z_3 z_4$ $\quad \vec{e}_{M2}=\left[z_2,z_1,-z_4,-z_3\right]^{\mathrm{T}}$

6. Classification of Main Operational Regimes

A. ***Whole Angle Standing Wave in Broad Sense (WA-SW-BS) Regime:***

$$S \to 0, \qquad N \to 0. \tag{46}$$

B. ***Whole Angle Standing Wave in the Proper Sense (WA-SW-PS) Regime:***

$$S \to 0, \qquad N \to 0, \qquad A \to 0. \tag{47}$$

C. ***Force-to-Rebalance in the Broad Sense (FTR-BS) Regime:***

$$S \to 0, \qquad RS \to 0, \qquad (\ 2RC+1 \neq 0\ ). \tag{48}$$

D. ***Force-to-Rebalance Standing Wave in the Proper Sense (FTR-SW-PS) Regime:***

$$S \to 0, \qquad RS \to 0, \qquad N \to 0. \tag{49}$$

E. ***Force-to-Rebalance Standing Wave in the Strict Sense (FTR-SW-SS) Regime:***

$$S \to 0, \qquad RS \to 0, \qquad N \to 0, \qquad AS \to 0. \tag{50}$$

F. ***Traveling Wave (TW) Regime:***

$$S \to 0, \qquad RS \to 0, \qquad RC \to 0. \tag{51}$$

7. Stabilization of the Main Operational Regimes

In what follows we will use the method of stabilization of the main operational regimes, proposed by V. Zhuravlev. That is why we will refer to the described methods as to the *Zhuravlev's autonomous stabilization methods.*

A. *Whole Angle Standing Wave in Broad Sense (WA-SW-BS) Regime*

This regime was originally considered by V. Zhuravlev. To ensure (46) we add the following stabilization force to the right part of equation (17):

$$\vec{F}_C\left(\vec{z},t\right)=-\vec{e}_S S-\vec{e}_N N\ . \tag{52}$$

It means that

$$\frac{dS}{dt}=\left(\frac{dS}{d\vec{z}}\cdot\frac{d\vec{z}}{dt}\right)=\left(\operatorname{grad} S\cdot F_C\right)=\vec{e}_S\left(-\vec{e}_S S-\vec{e}_N N\right)=-S\left(2S+1\right)-2N^2,$$

$$\frac{dN}{dt}=\left(\frac{dN}{d\vec{z}}\cdot\frac{d\vec{z}}{dt}\right)=\left(\operatorname{grad} N\cdot F_C\right)=\vec{e}_N\left(-\vec{e}_S S-\vec{e}_N N\right)=-N\left(4S+1\right)\ . \tag{53}$$

There are four equilibrium points ($\frac{dS}{dt}=\frac{dN}{dt}=0$) for equation (53) (Fig.3):

$$\text{1)}\quad S=0,\quad N=0\quad (\textit{stable node});$$

$$\text{2)}\quad S=-\frac{1}{2},\quad N=0\quad (\textit{unstable node});$$

$$\text{3, 4)}\quad S=-\frac{1}{4},\quad N=\pm\frac{1}{4}\quad (\textit{saddle points})\ . \tag{54}$$

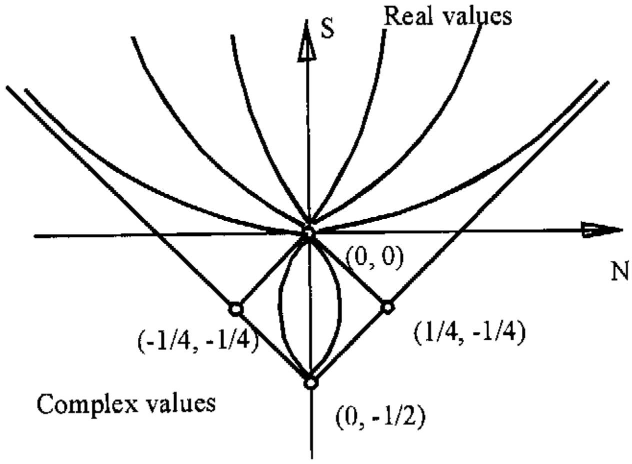

Fig.3.

Explicit solution of system (53) with initial conditions $S(t=0)=S_0, N(t=0)=N_0$ is:

$$S(t)=\frac{\left[S_0+2\left(S_0^2-N_0^2\right)\left(1-e^{-t}\right)\right]e^{-t}}{1+4S_0\left(1-e^{-t}\right)+4\left(S_0^2-N_0^2\right)\left(1-e^{-t}\right)^2}\ ,$$
$$N(t)=\frac{N_0\,e^{-t}}{1+4S_0\left(1-e^{-t}\right)+4\left(S_0^2-N_0^2\right)\left(1-e^{-t}\right)^2}\ . \tag{55}$$

For realization of this regime it is necessary to use the autonomous nonlinear combined forces (2) of the 13[th] *Class* with $k_{13}=4,\ \alpha_{13}=0$ and forces of negative damping:

$$\frac{d^2a}{dt^2}+a=\left[1-4\left(a^2+b^2\right)\right]\frac{da}{dt}\ ,$$
$$\frac{d^2b}{dt^2}+b=\left[1-4\left(a^2+b^2\right)\right]\frac{db}{dt}\ . \tag{56}$$

B. ***Whole Angle Standing Wave in the Proper Sense (WA-SW-PS) Regime***

Using the analogy with the *Case A*, we assume:

$$\vec{F}_C\left(\vec{z},t\right)=-\vec{e}_S S-\vec{e}_N N-\vec{e}_A A\ . \tag{57}$$

In this case, the dynamics of the gyro in terms of integral manifolds is described by the system:

$$\frac{dS}{dt}=-S(2S+1)-2\left(N^2+A^2\right);\qquad \frac{dN}{dt}=-N(4S+1);\qquad \frac{dA}{dt}=-A(4S+1)\ . \tag{58}$$

The equilibrium points and lines in this case are (Fig.4):

$$\begin{aligned}
&1)\quad S=0,\quad N=0,\quad A=0\quad (\textit{stable node})\ ;\\
&2)\quad S=-\frac{1}{2},\quad N=0,\quad A=0\quad (\textit{unstable node});\\
&3)\quad S=-\frac{1}{4},\quad N^2+A^2=\frac{1}{16}\quad (\textit{saddle circle})\ .
\end{aligned} \tag{59}$$

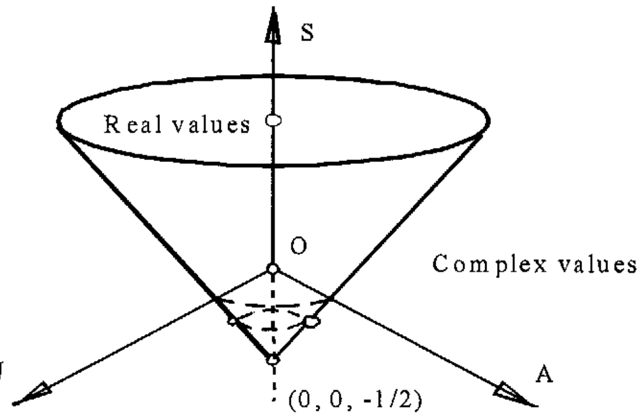

Fig.4.

Exact solution of the initial problem $S(t=0)=S_0$, $N(t=0)=N_0$, $A(t=0)=A_0$ could be obtained in a similar way as (55):

$$S(t)=\frac{\left[S_0+2\left(S_0^2-N_0^2-A_0^2\right)\left(1-e^{-t}\right)\right]e^{-t}}{1+4S_0\left(1-e^{-t}\right)+4\left(S_0^2-N_0^2-A_0^2\right)\left(1-e^{-t}\right)^2}\ ,$$

$$N(t)=\frac{N_0\,e^{-t}}{1+4S_0\left(1-e^{-t}\right)+4\left(S_0^2-N_0^2-A_0^2\right)\left(1-e^{-t}\right)^2}\ ,$$

$$A(t)=\frac{A_0\,e^{-t}}{1+4S_0\left(1-e^{-t}\right)+4\left(S_0^2-N_0^2-A_0^2\right)\left(1-e^{-t}\right)^2}\ . \tag{60}$$

C. ***Force-to-Rebalance in the Broad Sense (FTR-BS) Regime***

To stabilize regime (48) we apply the following correcting force:

$$\vec{F}_C(\vec{z},t)=-\vec{e}_S S-\vec{e}_R R\ . \tag{61}$$

The analysis is similar to (52)-(56).

D. ***Force-to-Rebalance Standing Wave in the Proper Sense (FTR-SW-PS) Regime***

This regime (49) is stabilized by force:

$$\vec{F}_C(\vec{z},t)=-\vec{e}_S S-\vec{e}_R R-\vec{e}_N N \tag{62}$$

and the analysis is similar to (57)-(59).

E. ***Force-to-Rebalance Standing Wave in the Strict Sense (FTR-SW-SS) Regime***

This regime (50) is stabilized by force:

$$\vec{F}_C(\vec{z},t) = -\vec{e}_S S - \vec{e}_R R - \vec{e}_N N - \vec{e}_A A \ . \tag{63}$$

In this case, the manifold's evolution is described by the system:

$$\frac{dS}{dt} = -S(2S+1) - 2\left(N^2 + A^2 + R^2\right), \qquad \frac{dN}{dt} = -N(4S+1) ,$$

$$\frac{dA}{dt} = -A(4S+1) - 2MR \ , \qquad \frac{dR}{dt} = -R(4S+1) - 2MA \ ; \tag{64}$$

where manifold $M = z_1 z_4 + z_2 z_3$ is not independent on S-, N-, A-, and R-manifolds:

$$\left[(2S+1)^2 - 4N^2\right]M^2 - 4AR(2S+1)M + \left\{4N^4 + \left[4\left(A^2+R^2\right) - (2S+1)^2\right]N^2 + 4A^2R^2\right\} = 0 \ . \tag{65}$$

F. ***Traveling Wave (TW) Regime***

Regime (51) is stabilized by the following control force:

$$\vec{F}_C(\vec{z},t) = -\vec{e}_S\, S - \vec{e}_{RS}\, RS - \vec{e}_{RC}\, RC \ , \tag{66}$$

and analysis is similar to (57)-(59).

8. Conclusions

1. It is shown that the main operational regimes of vibratory gyroscopes could be formulated in terms of the integral manifolds.
2. This approach substantially simplifies synthesis of the corresponding correction laws, which stabilize vibrating patterns in particular operational regimes.
3. The main advantage of the above-mentioned correction-stabilization laws is that they ensure the global stability of the operational regimes.
4. Qualitative behavior as well as the explicit solutions of nonlinear system of equations describing particular operational regimes is obtained.
5. The main results discussed in the present paper are applicable to a broad range of vibratory gyroscopes.