DYNAMICS OF ROTATING AND VIBRATING THIN HEMISPHERICAL SHELL WITH MASS AND DAMPING IMPERFECTIONS AND PARAMETRICALLY DRIVEN BY DISCRETE ELECTRODES

Michael Shatalov[1,2] and Charlotta Coetzee[2]

[1]Sensor Science and Technology (SST) of CSIR Material Science and Manufacturing (MSM),
CSIR, P.O. Box 395, Pretoria 0001, Republic of South Africa
E-mail: mshatlov@csir.co.za
[2]Department of Mathematics and Statistics
P.B.X680, Pretoria 0001 Tshwane University of Technology
(Технологический Университет Цване),
Republic of South Africa
E-mail: coetzeec@tut.co.za

Key words: Hemispherical resonator gyroscope, whole angle regime, parametric electrode, compensation of gyro drift.

Abstract

The parametric electrode is normally used in the hemispherical resonator gyroscopes, operating in the whole angle regime, for maintaining of amplitudes of vibratory patterns. It is well known that due to variation of the gap between the resonator and parametric electrode a drift of the vibrating pattern is obtained. This drift is similar to the gyro drift stipulated by the Q-factor imperfections and substantially deteriorates the quality of the hemispherical resonator gyroscope. In the present paper we consider the methods of compensation of these drifts by means of the special control of the vibrating pattern by the sectioned parametric electrode. Compensation of the drifts is achieved through amplitudes of voltage and phase manipulations at the sectioned parametric electrodes. The side effect of this control is discussed consisting of spurious splitting of frequencies of the vibrating pattern.

Introduction

Basics of the vibratory gyroscopes are given in [1] and [2]. The principles of operations of the hemispherical resonator gyroscopes with electric circuits are explained in [3]. The method of the quadrature signal suppression is described in [4]. In [5] the author considered the general ideology of the compensation of the difference in the Q-factor. In the present paper we give the detailed analysis of the electro-mechanical model of the gyroscope and show that functions of the amplitude stabilization and compensation of both the quadrature component and the Q-factor difference could be realized by means of the proper control of the discrete parametric drive.

Model of Thin Shell Resonator

Assume that a thin hemispherical shell with a fixed pole is subjected to simultaneous vibration and rotation about its axis of symmetry (Fig. 1). Angular rate of inertial rotation is supposed to be small comparison to the lowest eigenvalue $(\Omega << \omega_{\min})$. In this case all the dynamical effects proportional to the square of the inertial angular rate $(\Omega^2 \approx 0)$, such as centrifugal forces, could be neglected. The mid surface radius R and thickness of the shell h are supposed to be constant. A particular point on the mid surface of the shell is characterized by the pair of angles φ and θ. Mass density of the shell is not constant and for the sake of simplicity suppose that a function of the azimuthal angle φ is given by :

$$\rho(\varphi)=\rho_0+\varepsilon\sum_{n=1}^{\infty}\left(\rho_{nc}\cos n\varphi+\rho_{ns}\sin n\varphi\right) \tag{1}$$

where parameter ε is small ($0<\varepsilon<<1$).

A similar assumption is made for the damping coefficient distribution:

$$d(\varphi)=d_0+\varepsilon\sum_{n=1}^{\infty}\left(d_{nc}\cos n\varphi+d_{ns}\sin n\varphi\right). \tag{2}$$

Let a specific vibration, corresponding to the circumferential wave number m is sustained:

$$\begin{aligned} u_m(t,\theta,\varphi)&=X_m(\theta)\left[a(t)\cos m\varphi+b(t)\sin m\varphi\right]\\ v_m(t,\theta,\varphi)&=Y_m(\theta)\left[-a(t)\sin m\varphi+b(t)\cos m\varphi\right]\\ w_m(t,\theta,\varphi)&=Z_m(\theta)\left[a(t)\cos m\varphi+b(t)\sin m\varphi\right]\end{aligned} \tag{3}$$

where $X_m(\theta)$, $Y_m(\theta)$, $Z_m(\theta)$ - Rayleigh's inextensional deformations of the shell:

$$X=X_m(\theta)=-\sin\theta\tan^m\left(\frac{\theta}{2}\right),\quad Y=Y_m(\theta)=\sin\theta\tan^m\left(\frac{\theta}{2}\right),\quad Z=Z_m(\theta)=(m+\cos\theta)\tan^m\left(\frac{\theta}{2}\right) \tag{4}$$

It is supposed that the vibration is realized in the vicinity of one of eigenvalue (ω), corresponding to m, and that the effect of parametrical resonance is used for maintaining the amplitude of vibrations.

The amplitude of the vibration of the shell is maintained by means of the energy delivered through the electric field in gaps between the shell and a system of discrete electrodes surrounding the shell. The gaps between the shell and electrodes are constant and equal Δ.

There exist a constant difference of electric potentials between the shell and electrodes and exiting voltage has the form of meander and varies with double frequency of the shell vibrations. By means of this driving the effect of parametric resonance excitation is achieved.

Kinetic, Strain, and Electric Energies of the System

The kinetic energy of the shell is given by

$$\begin{aligned} T^{(shell)}=&\frac{hR^2}{2}\int_0^{2\pi}\int_0^{\pi/2}\rho(\varphi)V^2\sin\theta\,d\theta\,d\varphi=\left\{\frac{\pi\rho_0hR^2}{2}\int_0^{\pi/2}\left[X_m^2(\theta)+Y_m^2(\theta)+Z_m^2(\theta)\right]\sin\theta\,d\theta\right\}\left[\dot a^2(t)+\dot b^2(t)\right]\\ &+\left\{\frac{\pi hR^2}{4}\int_0^{\pi/2}\left[X_m^2(\theta)-Y_m^2(\theta)+Z_m^2(\theta)\right]\sin\theta\,d\theta\right\}\left\{\rho_{2mc}\left[\dot a^2(t)-\dot b^2(t)\right]+2\rho_{2ms}\left[\dot a(t)\dot b(t)\right]\right\}\\ &-\left\{2\pi\rho_0hR^2\cdot\Omega\cdot\int_0^{\pi/2}\left[X_m(\theta)\cos\theta+Z_m(\theta)\sin\theta\right]Y_m(\theta)\sin\theta\,d\theta\right\}\left[\dot a(t)b(t)-a(t)\dot b(t)\right]\end{aligned} \tag{5}$$

where angles φ and θ define the location of an element on a mid surface of the shell.

In the context of the Gol'denveyzer-Novozhilov theory of thin isotropic shells subjected to inextensional deformations, the strain energy is as follows:

$$P=\frac{Eh^3R^2}{24\left(1-\eta^2\right)}\int_0^{2\pi}\int_0^{\theta_0}\left[\kappa_1^2+\kappa_1^2+2\eta\,\kappa_1\kappa_2+2\left(1-\eta\right)\tau^2\right]\sin\theta\,d\theta\,d\varphi \tag{6}$$

where E is the modulus of elasticity, η Poisson's ratio, κ_1, κ_2, and τ are strain components in the Gol'denveyzer-Novozhilov theory, which are, by using expressions (3) in the case of a spherical shell, given by:

$$\kappa_1=\frac{1}{R^2}\left(\frac{dX}{d\theta}-\frac{d^2Z}{d\theta^2}\right)\left(a\cos m\varphi+b\sin m\varphi\right),$$

$$\kappa_2=\frac{1}{R^2}\left(\cot\theta\,X-\frac{m}{\sin\theta}Y+\frac{m^2}{\sin^2\theta}Z-\cot\theta\frac{dZ}{d\theta}\right)\left(a\cos m\varphi+b\sin m\varphi\right), \tag{7}$$

$$\tau=\frac{1}{R^2}\left(\frac{m}{\sin\theta}X-\cot\theta\,Y+\frac{m\cot\theta}{\sin\theta}Z+\frac{dY}{d\theta}-\frac{m}{\sin\theta}\frac{dZ}{d\theta}\right)\left(-a\sin m\varphi+b\cos m\varphi\right)$$

(we omit indexes "m" for the sake of simplification of the notation).

Finally, the strain energy of the hemispherical shell (6) is as follows:

$$P^{(hem)}=\left\{\frac{\pi m^2\left(m^2-1\right)^2Eh^3}{6(1+\eta)R^2}\cdot\int_0^{\pi/2}\frac{\tan^{2m}\left(\frac{\theta}{2}\right)}{\sin^3\theta}d\theta\right\}\cdot\left[a^2(t)+b^2(t)\right]. \tag{8}$$

For derivation of the electric energy of the system we assume that the system of N ($N=8\cdot m$, where m - circumferential wave number) electrodes surrounding the shell and equidistantly located from the outer side. Every electrode has an angular measure in φ - direction equals to $2\Delta\varphi$. Hence, the centres of the electrodes are located at the angles 0, $2\pi/N$, $4\pi/N$, ... , $2\pi(N-1)/N$. In the θ - direction, the electrodes are restricted by angles θ_1 and θ_2 ($0<\theta_1\le\theta\le\theta_2\le\pi/2$). Let $U_i\,(t,\varphi)$ be the difference of electric potentials between the shell and the i - electrode. Gaps between the shell and electrodes are homogeneous and equal Δ and the quasi-electrostatic field is concentrated in these gaps. The electric energy of the quasi-electrostatic field is described by the expression:

$$\begin{aligned}W=\frac{\varepsilon_0\bar{R}^2}{2}\sum_{i=1}^{N}\int_{\varphi_{1i}}^{\varphi_{2i}}\int_{\theta_1}^{\theta_2}\frac{U_i^2(t,\varphi)}{\Delta-w}\sin\theta\,d\theta\,d\varphi=\frac{\varepsilon_0\bar{R}^2}{\Delta^3}\left[\int_{\theta_1}^{\theta_2}Z_m^2(\theta)\,\sin\theta\,d\theta\right]&\left\{2m\Delta\varphi\left(V_1^2+V_2^2+V_3^2+V_4^2\right)\frac{a^2(t)+b^2(t)}{2}\right.\\
+\frac{8m\Delta\varphi}{\pi}\left(\Delta V_1^2+\Delta V_2^2+\Delta V_3^2+\Delta V_4^2\right)\cdot\frac{a^2(t)+b^2(t)}{2}\cdot\sin(2\omega t)&\\
+\sin(2m\Delta\varphi)\left[\left(V_1^2-V_3^2\right)\frac{a^2(t)-b^2(t)}{2}+\left(V_2^2-V_4^2\right)a(t)b(t)\right]&\\
\left.+\frac{4\sin(2m\Delta\varphi)}{\pi}\left[\left(\Delta V_1^2-\Delta V_3^2\right)\frac{a^2(t)-b^2(t)}{2}+\left(\Delta V_2^2-\Delta V_4^2\right)a(t)b(t)\right]\sin(2\omega t)\right\}&\end{aligned} \tag{9}$$

where $\varphi_{2i}-\varphi_{1i}=2\Delta\varphi$, ε_0 is the absolute di-electric permittivity, $\bar{R}=R+1/2(h+\Delta)$, and $U_i\,(t,\varphi)=U_{0i}(\varphi)+\Delta U_i(t,\varphi)$ the difference of the electric potentials between the shell and the i -

electrode, where $U_{0i}(\varphi)$ is the time independent component of the voltage, and $\Delta U_i(t,\varphi)$ is the time dependent meander-shaped signal with half-period compare to the shell frequency (parametric excitation). It is more convenient to consider the square of $U_i^2(t,\varphi)=V_{0i}^2(\varphi)+\Delta V_i^2(t,\varphi)$, where the following notation is used: $V_{0i}^2(t,\varphi)=U_{0i}^2(\varphi)+\Delta U_i^2(t,\varphi)$, $\Delta V_i^2(t,\varphi)=2\cdot U_{0i}(\varphi)\cdot\Delta U_i(t,\varphi)$. Hence, $V_{0i}^2(\varphi)\geq\Delta V_i^2(t,\varphi)$, and equality is possible in the special case if $U_{0i}(\varphi)$ equals the amplitude of $\Delta U_i(t,\varphi)$. Furthermore it is supposed that $N=8\cdot m$ electrodes are combined in four groups of $2\cdot m$ electrodes and each group form a $2\cdot m$ regular polygon. Furthermore, assume that the voltages applied to every electrode of a particular group are the same and equal to V_j, where $j=1,2,3,4$ is the number of the corresponding group:

- For the first group ($\varphi_1=0$, $\varphi_2=\pi/m$, $\varphi_2=2\pi/m$, ... , $\varphi_{2m}=(2m-1)\pi/m$):
$$\sum_{i=1}^{2m}\left[V_i^2(\varphi)+\Delta V_i^2(t,\varphi)\right]=\left\{V_1^2+\frac{4\Delta V_1^2}{\pi}\sin(2\omega t)\right\}\left\{\frac{2m\Delta\varphi}{\pi}+\frac{2}{\pi}\sin(2m\Delta\varphi)\cos 2m\varphi\right\},$$
- For the second group ($\varphi_{2m+1}=\pi/4m$, $\varphi_2=\pi/4m+\pi/m$, $\varphi_2=\pi/4m+2\pi/m$, ... , $\varphi_{2m}=\pi/4m+(2m-1)\pi/m$):
$$\sum_{i=2m+1}^{4m}\left[V_i^2(\varphi)+\Delta V_i^2(t,\varphi)\right]=\left\{V_2^2+\frac{4\Delta V_2^2}{\pi}\sin(2\omega t)\right\}\left\{\frac{2m\Delta\varphi}{\pi}+\frac{2}{\pi}\sin(2m\Delta\varphi)\sin 2m\varphi\right\},$$
- For the third group ($\varphi_{2m+1}=\pi/2m$, $\varphi_2=\pi/2m+\pi/m$, $\varphi_2=\pi/2m+2\pi/m$, ... , $\varphi_{2m}=\pi/2m+(2m-1)\pi/m$):
$$\sum_{i=4m+1}^{6m}\left[V_i^2(\varphi)+\Delta V_i^2(t,\varphi)\right]=\left\{V_3^2+\frac{4\Delta V_3^2}{\pi}\sin(2\omega t)\right\}\left\{\frac{2m\Delta\varphi}{\pi}-\frac{2}{\pi}\sin(2m\Delta\varphi)\cos 2m\varphi\right\},$$
- For the fourth group ($\varphi_{2m+1}=3\pi/4m$, $\varphi_2=3\pi/4m+\pi/m$, $\varphi_2=3\pi/4m+2\pi/m$, ... , $\varphi_{2m}=3\pi/4m+(2m-1)\pi/m$):
$$\sum_{i=6m+1}^{8m}\left[V_i^2(\varphi)+\Delta V_i^2(t,\varphi)\right]=\left\{V_4^2+\frac{4\Delta V_4^2}{\pi}\sin(2\omega t)\right\}\left\{\frac{2m\Delta\varphi}{\pi}-\frac{2}{\pi}\sin(2m\Delta\varphi)\sin 2m\varphi\right\}.$$

Rayleigh Dissipative Function

Damping effects of the shell are introduced by means of the Rayleigh dissipative function:

$$D=\frac{hR^2}{2}\int_0^{2\pi}\int_0^{\pi/2}d(\varphi)\left(\dot{u}^2+\dot{v}^2+\dot{w}^2\right)\sin\theta\,d\theta\,d\varphi=\left\{\pi hR^2 d_0\int_0^{\pi/2}\left[X_m^2(\theta)+Y_m^2(\theta)+Z_m^2(\theta)\right]\sin\theta\,d\theta\right\}\frac{\dot{a}^2(t)+\dot{b}^2(t)}{2}$$
$$+\left\{\frac{\pi hR^2}{2}\int_0^{\pi/2}\left[X_m^2(\theta)-Y_m^2(\theta)+Z_m^2(\theta)\right]\sin\theta\,d\theta\right\}\left\{d_{2mc}\frac{\dot{a}^2(t)-\dot{b}^2(t)}{2}+d_{2ms}\dot{a}(t)\dot{b}(t)\right\} \quad (10)$$

Lagrangian of the System and Equations of the Vibrating Pattern

The Lagrangian of the system is as follows:

$$L = T - P + W = L\left(\dot{a}.\dot{b}, a, b\right) \tag{11}$$

where kinetic energy (T), potential energy (P), and electric energy (W) are given by expressions (8), (8), and (9) correspondingly.

The Euler-Lagrange equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right) - \frac{\partial L}{\partial a} = -\frac{\partial D}{\partial \dot{a}}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{b}}\right) - \frac{\partial L}{\partial b} = -\frac{\partial D}{\partial \dot{b}} \tag{12}$$

where the Rayleigh dissipative function is given by (10).

Substituting the corresponding expressions for energies and dissipative function in equations (12) we obtain the following system of equations:

$$\begin{gathered}\left(1+\varepsilon\overline{\Delta}_{1C}\right)\ddot{a} + \varepsilon\overline{\Delta}_{1S}\ddot{b} + 2\varepsilon\delta\left(1+\overline{\Delta}_{2C}\right)\dot{a} + 2\varepsilon\delta\overline{\Delta}_{2S}\dot{b} + \omega^2\left(1-\varepsilon\overline{\Delta}_{3C}\right)a - \omega^2\varepsilon\overline{\Delta}_{3S}b - 2\varepsilon\eta\overline{\Omega}\dot{b} \\ -\varepsilon p_0 a\sin(2\omega t) - \varepsilon p_{2C} a\sin(2\omega t) - \varepsilon p_{2S} b\sin(2\omega t) = 0, \\ \varepsilon\overline{\Delta}_{1S}\ddot{a} + \left(1-\varepsilon\overline{\Delta}_{1C}\right)\ddot{b} + 2\varepsilon\delta\overline{\Delta}_{2S}\dot{a} + 2\varepsilon\delta\left(1-\overline{\Delta}_{2C}\right)\dot{b} - \omega^2\varepsilon\overline{\Delta}_{3S}a + \omega^2\left(1+\varepsilon\overline{\Delta}_{3C}\right)b + 2\varepsilon\eta\overline{\Omega}\dot{a} \\ -\varepsilon p_0 b\sin(2\omega t) - \varepsilon p_{2S} a\sin(2\omega t) + \varepsilon p_{2C} b\sin(2\omega t) = 0\end{gathered} \tag{13}$$

where ε is a small parameter characterizing "smallness" of the corresponding effects in the system and $\varepsilon\overline{\Delta}_{1C} = \dfrac{\rho_{2mc}}{\rho_0}\dfrac{I_1}{I_0}$, $\varepsilon\overline{\Delta}_{1S} = \dfrac{\rho_{2ms}}{\rho_0}\dfrac{I_1}{I_0}$, $\varepsilon\eta\overline{\Omega} = \dfrac{2I_2}{I_0}\Omega$, $2\varepsilon\delta = \dfrac{d_{2mc}}{d_0}$, $2\varepsilon\delta\Delta_{2C} = \dfrac{d_{2mc}}{2\rho_0}\dfrac{I_1}{I_0}$, $2\varepsilon\delta\Delta_{2C} = \dfrac{d_{2mc}}{2\rho_0}\dfrac{I_1}{I_0}$,

$\omega^2 = \dfrac{m^2\left(m^2-1\right)^2 h^2}{R^4}\dfrac{E}{(1+\eta)\rho_0}\dfrac{I_3}{I_0} - \dfrac{\sigma_1}{\pi h R^2\rho_0}\dfrac{I_4}{I_0}$, $\omega^2\varepsilon\Delta_{3C} = \dfrac{\sigma_2}{\pi h R^2\rho_0}\dfrac{I_4}{I_0}$,

$\omega^2\varepsilon\Delta_{3S} = \dfrac{\sigma_3}{\pi h R^2\rho_0}\dfrac{I_4}{I_0}$, $\varepsilon p_0 = \dfrac{\Delta\sigma_1}{\pi h R^2\rho_0}\dfrac{I_4}{I_0}$, $\varepsilon p_{2C} = \dfrac{\Delta\sigma_2}{\pi h R^2\rho_0}\dfrac{I_4}{I_0}$, $\varepsilon p_{2S} = \dfrac{\Delta\sigma_3}{\pi h R^2\rho_0}\dfrac{I_4}{I_0}$,

$I_1 = \dfrac{1}{2}\displaystyle\int_0^{\theta_0}\left[X_m^2(\theta) - Y_m^2(\theta) + Z_m^2(\theta)\right]\sin\theta\, d\theta$, $I_2 = \displaystyle\int_0^{\theta_0}\left[X_m(\theta)\cos\theta + Z_m(\theta)\sin\theta\right]Y_m(\theta)\sin\theta\, d\theta$,

$I_3 = \dfrac{1}{2}\displaystyle\int_0^{\theta_0}\dfrac{\tan^{2m}\left(\frac{\theta}{2}\right)}{\sin^3\theta}d\theta$, $\quad I_4 = \displaystyle\int_0^{\theta_0} Z_m^2(\theta)\sin\theta\, d\theta$, $\sigma_1 = \left(V_1^2 + V_2^2 + V_3^2 + V_4^2\right)\dfrac{\varepsilon_0\overline{R}^2}{\Delta^3}2m\Delta\varphi$,

$\sigma_2 = \left(V_1^2 - V_3^2\right)\dfrac{\varepsilon_0\overline{R}^2}{\Delta^3}\sin(2m\Delta\varphi)$, $\sigma_3 = \left(V_2^2 - V_4^2\right)\dfrac{\varepsilon_0\overline{R}^2}{\Delta^3}\sin(2m\Delta\varphi)$,

$\Delta\sigma_1 = \left(\Delta V_1^2 + \Delta V_2^2 + \Delta V_3^2 + \Delta V_4^2\right)\dfrac{\varepsilon_0\overline{R}^2}{\Delta^3}\dfrac{8m\Delta\varphi}{\pi}$, $\qquad \Delta\sigma_2 = \left(\Delta V_1^2 - \Delta V_3^2\right)\dfrac{\varepsilon_0\overline{R}^2}{\Delta^3}\sin(2m\Delta\varphi)$,

$\Delta\sigma_3 = \left(\Delta V_2^2 - \Delta V_4^2\right)\dfrac{\varepsilon_0\overline{R}^2}{\Delta^3}\sin(2m\Delta\varphi)$.

Solving the equations of system (13) with respect to $\ddot{a}$, $\ddot{b}$ and neglecting terms of order $O\left(\varepsilon^2\right)$ we obtain the following system:

$$\ddot{a}+\omega^2 a=\varepsilon F_1\left(\dot{a},\dot{b},a,b\right), \qquad \ddot{b}+\omega^2 b=\varepsilon F_1\left(\dot{a},\dot{b},a,b\right) \tag{14}$$

where

$$\begin{aligned} F_1\left(\dot{a},\dot{b},a,b\right) &= 2\eta\bar{\Omega}\dot{b}-2\delta\left(1+\Delta_{2C}\right)\dot{a}-2\delta\Delta_{2S}\dot{b}+\omega^2\Delta_{1C}a+\omega^2\Delta_{1S}b \\ &\quad +p_0 a\sin\left(2\omega t\right)+p_{2C}a\sin\left(2\omega t\right)+p_{2S}b\sin\left(2\omega t\right) \\ F_2\left(\dot{a},\dot{b},a,b\right) &= -2\eta\bar{\Omega}\dot{a}-2\delta\Delta_{2S}\dot{a}-2\delta\left(1-\Delta_{2C}\right)\dot{b}+\omega^2\Delta_{1S}a-\omega^2\Delta_{1C}b \\ &\quad +p_0 b\sin\left(2\omega t\right)+p_{2S}a\sin\left(2\omega t\right)-p_{2C}b\sin\left(2\omega t\right) \end{aligned} \tag{15}$$

where $\Delta_{2C}=\bar{\Delta}_{2C}$, $\Delta_{2S}=\bar{\Delta}_{2S}$, $\Delta_{1C}=\bar{\Delta}_{1C}+\bar{\Delta}_{3C}$, $\Delta_{1S}=\bar{\Delta}_{1S}+\bar{\Delta}_{3S}$.

Let us represent the vibrating pattern $\left(Z\right)$ of the mode with circumferential wave number m as follows:

$$Z=a\cos m\varphi+b\sin m\varphi=P\cos m\left(\varphi-\theta\right)\sin\left(\zeta-\psi\right)+Q\sin m\left(\varphi-\theta\right)\cos\left(\zeta-\psi\right) \tag{15}$$

where $\zeta=\omega t$. It follows from this representation that

$$a=P\cos m\theta\sin\left(\zeta-\psi\right)-Q\sin m\theta\cos\left(\zeta-\psi\right), \quad b=P\sin m\theta\sin\left(\zeta-\psi\right)+Q\cos m\theta\cos\left(\zeta-\psi\right), \tag{16}$$

$$\dot{a}=\omega\left[P\cos m\theta\cos\left(\zeta-\psi\right)+Q\sin m\theta\sin\left(\zeta-\psi\right)\right], \quad \dot{b}=\omega\left[P\sin m\theta\cos\left(\zeta-\psi\right)-Q\cos m\theta\sin\left(\zeta-\psi\right)\right]$$

It follows from (36|) that

$$\begin{aligned} \ddot{a}=\omega\Big\{&\dot{P}\cos m\theta\cos\left(\zeta-\psi\right)+\dot{Q}\sin m\theta\sin\left(\zeta-\psi\right)+m\dot{\theta}\left[-P\sin m\theta\cos\left(\zeta-\psi\right)+Q\cos m\theta\sin\left(\zeta-\psi\right)\right] \\ &+\left(\omega-\dot{\psi}\right)\left[-P\cos m\theta\sin\left(\zeta-\psi\right)+Q\sin m\theta\cos\left(\zeta-\psi\right)\right]\Big\}, \\ \ddot{b}=\omega\Big\{&\dot{P}\sin m\theta\cos\left(\zeta-\psi\right)-\dot{Q}\cos m\theta\sin\left(\zeta-\psi\right)+m\dot{\theta}\left[P\cos m\theta\cos\left(\zeta-\psi\right)+Q\sin m\theta\sin\left(\zeta-\psi\right)\right] \\ &+\left(\omega-\dot{\psi}\right)\left[-P\sin m\theta\sin\left(\zeta-\psi\right)-Q\cos m\theta\cos\left(\zeta-\psi\right)\right]\Big\} \end{aligned} \tag{17}$$

Substituting (16) – (17) into (14) and solving the resulting system with respect to $\dot{P},\dot{Q},m\dot{\theta}$, and $\dot{\psi}$ we obtain:

$$\dot{P}=\frac{\varepsilon}{\omega}\left[\overline{F_1}\cos m\theta+\overline{F_2}\sin m\theta\right]\cos\left(\zeta-\psi\right), \qquad \dot{Q}=\frac{\varepsilon}{\omega}\left[\overline{F_1}\sin m\theta-\overline{F_2}\cos m\theta\right]\sin\left(\zeta-\psi\right), \tag{18}$$

$$m\dot{\theta}=\frac{\varepsilon}{\omega\left(P^2-Q^2\right)}\left\{-\left[\overline{F_1}\sin m\theta-\overline{F_2}\cos m\theta\right]P\cos\left(\zeta-\psi\right)-\left[\overline{F_1}\cos m\theta+\overline{F_2}\sin m\theta\right]Q\sin\left(\zeta-\psi\right)\right\},$$

$$\dot{\psi}=\frac{\varepsilon}{\omega\left(P^2-Q^2\right)}\left\{\left[\overline{F_1}\sin m\theta-\overline{F_2}\cos m\theta\right]Q\cos\left(\zeta-\psi\right)+\left[\overline{F_1}\cos m\theta+\overline{F_2}\sin m\theta\right]P\sin\left(\zeta-\psi\right)\right\}$$

where $\overline{F_{1,2}}=\overline{F_{1,2}}\left(P,Q,\theta,\psi\right)$ is the result of substitution (16) into (15).

Solution of the system by using the method of averaging

Let us use the method of averaging for the solution of system (18). This method is based on a representation of the right hand side of the equation by functions which are averaged with respect to the "fast variable" $\zeta=\omega t$:

$$\left\langle\dot{P}\right\rangle=\frac{1}{2\pi}\int_0^{2\pi}\left(\dot{P}\right)d\zeta, \quad \left\langle\dot{Q}\right\rangle=\frac{1}{2\pi}\int_0^{2\pi}\left(\dot{Q}\right)d\zeta, \quad \left\langle\dot{\theta}\right\rangle=\frac{1}{2\pi}\int_0^{2\pi}\left(\dot{\theta}\right)d\zeta, \quad \left\langle\dot{\psi}\right\rangle=\frac{1}{2\pi}\int_0^{2\pi}\left(\dot{\psi}\right)d\zeta \tag{19}$$

The result of the averaging of the right hand side of the principal amplitude derivative of the vibrating pattern is as follows:

$$\left\langle \dot{P} \right\rangle = \varepsilon \left\{ -\delta P\left[1+\Delta_{2C}\cos 2m\theta + \Delta_{2S}\sin 2m\theta\right] - \frac{\omega Q}{2}\left[\Delta_{1C}\sin 2m\theta - \Delta_{1S}\cos 2m\theta\right] \right. \tag{20}$$

$$\left. + \frac{p_0}{4\omega} P\cos 2\psi + \frac{p_{2C}}{4\omega}\left[P\cos 2m\theta\cos 2\psi - Q\sin 2m\theta\sin 2\psi\right] + \frac{p_{2S}}{4\omega}\left[P\sin 2m\theta\cos 2\psi + Q\cos 2m\theta\sin 2\psi\right] \right\}$$

After averaging, the dynamics of the quadrature amplitude of the vibrating pattern is:

$$\left\langle \dot{Q} \right\rangle = \varepsilon \left\{ -\delta Q\left[1-\Delta_{2C}\cos 2m\theta - \Delta_{2S}\sin 2m\theta\right] + \frac{\omega P}{2}\left[\Delta_{1C}\sin 2m\theta - \Delta_{1S}\cos 2m\theta\right] \right. \tag{21}$$

$$\left. - \frac{p_0}{4\omega} Q\cos 2\psi + \frac{p_{2C}}{4\omega}\left[-P\sin 2m\theta\sin 2\psi + Q\cos 2m\theta\cos 2\psi\right] + \frac{p_{2S}}{4\omega}\left[P\cos 2m\theta\sin 2\psi + Q\sin 2m\theta\cos 2\psi\right] \right\}$$

The result of averaging of the precession rate of the vibrating pattern is as follows:

$$\left\langle m\dot{\theta} \right\rangle = \varepsilon \left\{ -\eta\,\bar{\Omega} + \delta\frac{P^2+Q^2}{P^2-Q^2}\left[\Delta_{2C}\sin 2m\theta - \Delta_{2S}\cos 2m\theta\right] - \omega\frac{PQ}{P^2-Q^2}\left[\Delta_{1C}\cos 2m\theta + \Delta_{1S}\sin 2m\theta\right] \right.$$

$$\left. + \frac{p_0}{2\omega}\frac{PQ}{P^2-Q^2}\sin 2\psi + \frac{1}{4\omega}\left[-p_{2C}\sin 2m\theta + p_{2S}\cos 2m\theta\right]\cos 2\psi \right\}. \tag{22}$$

Finally the dynamics of the phase shift of the vibrating pattern is described by the following averaging expression:

$$\left\langle \dot{\psi} \right\rangle = \varepsilon \left\{ -\delta\frac{2PQ}{P^2-Q^2}\left[\Delta_{2C}\sin 2m\theta - \Delta_{2S}\cos 2m\theta\right] + \frac{\omega}{2}\frac{P^2+Q^2}{P^2-Q^2}\left[\Delta_{1C}\cos 2m\theta + \Delta_{1S}\sin 2m\theta\right] \right.$$

$$\left. - \frac{1}{4\omega}\left[p_0\frac{P^2+Q^2}{P^2-Q^2} + \frac{p_{2C}}{4\omega}\cos 2m\theta + \frac{p_{2S}}{4\omega}\sin 2m\theta \right]\sin 2\psi \right\}. \tag{23}$$

Let us assume that the gyro control system guarantee smallness of the quadrature amplitude (Q) and the phase shift (ψ). In this case equations (20) – (23) could be simplified to:

$$\left\langle \dot{P} \right\rangle \approx \varepsilon P \left\{ -\delta\left[1+\Delta_{2C}\cos 2m\theta + \Delta_{2S}\sin 2m\theta\right] + \frac{1}{4\omega}\left[p_0 + p_{2C}\cos 2m\theta + p_{2S}\sin 2m\theta\right] \right\},$$

$$\left\langle \dot{Q} \right\rangle \approx \varepsilon \left\{ \frac{\omega P}{2}\left[\Delta_{1C}\sin 2m\theta - \Delta_{1S}\cos 2m\theta\right] \right\},$$

$$\left\langle m\dot{\theta} \right\rangle \approx \varepsilon \left\{ -\eta\,\bar{\Omega} + \delta\left[\Delta_{2C}\sin 2m\theta - \Delta_{2S}\cos 2m\theta\right] + \frac{1}{4\omega}\left[-p_{2C}\sin 2m\theta + p_{2S}\cos 2m\theta\right] \right\},$$

$$\left\langle \dot{\psi} \right\rangle \approx \varepsilon \left\{ \frac{\omega}{2}\left[\Delta_{1C}\cos 2m\theta + \Delta_{1S}\sin 2m\theta\right] \right\}. \tag{24}$$

It follows from the first equation of (24) that the damping effects of the resonator could be compensated by means of proper control of the vibrating pattern: angle (θ), is the independent decay of the principal amplitude and is compensated by the averaged parametric excitation, hence $-\delta + \frac{p_0}{4\omega} \to 0$. Furthermore the angle dependent decay of the principal amplitude is compensated by the asymmetric parametric excitation, i.e. $-\delta\Delta_{2C}+\dfrac{p_{2C}}{4\omega}\to 0$ and $-\delta\Delta_{2S}+\dfrac{p_{2S}}{4\omega}\to 0$. Keep in mind that in this case the angle-dependent drift is compensated as it follows from the third equation of (24) and $\left\langle m\dot{\theta}\right\rangle\to-\varepsilon\eta\bar{\Omega}$ which corresponds to the pure Bryan effect. For the suppression of the quadrature component (Q), it is possible to manipulate with the components $\bar{\Delta}_{3C}$, $\bar{\Delta}_{3S}$ so that

$$\bar{\Delta}_{3C}\sim\sigma_2\sim\left(\Delta V_1^2-\Delta V_3^2\right)\sim -Q\sin\left(2m\theta\right),\qquad \bar{\Delta}_{3S}\sim\sigma_3\sim\left(\Delta V_2^2-\Delta V_4^2\right)\sim Q\cos\left(2m\theta\right) \quad (25)$$

and hence,

$$\bar{\Delta}_{3C}=-kQ\sin\left(2m\theta\right),\qquad \bar{\Delta}_{3S}=kQ\cos\left(2m\theta\right) \quad (26)$$

where k is coefficient of proportionality. In this case, as it follows from (21) and (24):

$$\left\langle\dot{Q}\right\rangle\approx-\varepsilon\left(\frac{p_0}{2\omega}+k\frac{\omega P}{2}\right)Q\to 0 \quad (27)$$

All other terms in (21) could be considered as perturbing terms.
For the realization of the control $\psi\to 0$ it is possible to use a reference phase generator which generates the reference excitation signals so that $\psi\approx 0$.

Conclusions

The electromechanical model of the hemispherical resonator gyroscope is considered. It is shown that the main control functions of maintaining the vibration amplitude, compensation of the quadrature signal, and the suppression of the gyro drift due to the difference in the Q-factor could be realized by the discrete electrodes parametric drive.

References

1. Zhuravlev V. Ph., Klimov D. M., Hemispherical resonator gyroscope, Nauka, Moscow, 1985 (in Russian).
2. Lynch D.D., Vibratory gyro Analysis by the method of averaging, II St. Petersburg International Conference on Integrated Navigation Systems, Part II, 1995, St. Petersburg, pp.18-26.
3. Zhuravlev V. Ph., Lynch D.D., Electric model of a hemispherical resonator gyroscope, Mechanics of Solids, 1995, Vol. 30 (5), pp.10-21.
4. Zhbanov Yu. K., Self-tuning contour of suppressing of quadrature of a hemispherical resonator gyroscope, Giroskopiya I Navigatsiya, 2007, No.2, pp.37-42.
5. Zhbanov Yu. K., Amplitude control contour in a hemispherical resonator gyro with automatic compensation for difference in Q-factor, Mechanics of Solids, 2008, Vol. 40, No. 3, pp. 328-332.