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COMPARISON OF CLASSICAL AND MODERN THEORIES OF LONGITUDINAL WAVE PROPAGATION IN ELASTIC RODS

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A unified approach to derivation of different families of differential equations describing the longitudinal vibration of elastic rods and based on the Hamilton variational principle is outlined. The simplest model of longitudinal vibration of the rods does not take into consideration its lateral motion and is described in terms of the wave equation. The more elaborated models were proposed by Rayleigh, Love, Bishop, Mindlin-Herrmann, and multimode models in which the lateral effect plays an important role. Dispersion curves, representing the eigenvalues versus wave numbers, of these models are compared with the exact dispersion curves of isotropic cylinder and conclusions on accuracy of the models are deduced. The Green functions are constructed for the classical, Rayleigh, Bishop, and Mindlin-Herrmann models in which the general solutions of the problem are obtained. The principles of construction of the multimode theories, corresponding equations and orthogonality conditions are considered.

1. Introduction

In what follow, the wave displacements in the rod are described in accordance with the assumptions made in various vibration theories; afterwards the Hamilton variational principle is used to derive the equation or system of equation of motion corresponding to each approach. The method of finding the analytical solution is based on the separation of variables, the investigation of the eigenfunctions from the Sturm-Liouville problem, proof of two kinds of the eigenfunction orthogonality conditions by using the equations of the Sturm-Liouville problem. At the next stage the solution is assumed in the form of the Fourier series and substituted into the Lagrangian which hold the Euler-Lagrange differential equation. The solution of the resulting differential equation is used together with the norms corresponding to the above orthogonalities to construct the Green function. The general solutions of the problems are formulated in terms of the Green functions.

2. Classical theory: wave equation

In this case the longitudinal displacement is assumed constant in all points along the cross section of the rod and expressed as $u = u(x,t)$. The general or compact form of equation of motion is:

$$\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u'}\right) - \frac{\partial \Lambda}{\partial u} = 0 \tag{1}$$

with the natural boundary conditions $\left.\frac{\partial \Lambda}{\partial u'}\right|_{x=0,l} = 0$ or $u(x,t)\big|_{x=0,l} = 0$, where

$$\Lambda = \Lambda(\dot{u}, u', u) = \frac{1}{2}\left[A(x)\left(\rho(x)\dot{u}^2 - E(x)u'^2\right)\right] + A(x)F(x,t)u \tag{2}$$

is the Lagrangian density in which: $A(x)$ is the cross-sectional area of the rod, $\rho(x)$ is the mass density of the rod $E(x)$ is the Young modulus of elasticity and $F(x,t)$ is the applied external force.

Substituting (2) into (1) we obtain the explicit form of the equation (1):

$$A(x)\rho(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(A(x)E(x)\frac{\partial u}{\partial x}\right) - A(x)F(x,t) = 0 \tag{3}$$

The eigenfunction follow from the corresponding sturm-Liouville problem fulfilled two orthogonality conditions:

$$\int_0^l X_n(x)X_m(x)\,dx = 0 \qquad \text{and} \qquad \int_0^l X'_n(x)X'_m(x)\,dx = 0 \qquad \text{for} \qquad m \neq n \tag{4}$$

where $\|X_n\|_1^2 = \int_0^l X_x^2 dx$ and $\|X_n\|_2^2 = \int_0^l X_n'^2 dx$, $X_n(x)$, $n = 1, 2, ...$ are the eigenfunctions corresponding to the eigenvalues $\Omega_{1,n} = \sqrt{\frac{E}{\rho}}\frac{\|X_n\|_2}{\|X_n\|_1}$.

The solution of the problem is given by the following expression:

$$u(x,t) = \int_0^l g(\xi)\frac{\partial G(x,\xi,t)}{\partial t}d\xi + \int_0^l h(\xi)G(x,\xi,t)\,d\xi + \frac{1}{\rho}\int_0^t\int_0^l F(\xi,\tau)G(x,\xi,t-\tau)\,d\xi\,d\tau \tag{5}$$

where $G_1(x,\xi,t) = \sum_{n=1}^{\infty}\frac{X_n(x)X_n(\xi)\sin\Omega_{1,n}t}{\Omega_{1,n}\|X_n\|_1^2}$ is the Green function.

3. Rayleigh-Love theory

The effects of the lateral displacement of the rod are taken into consideration in the kinetic energy by introducing the Poisson ratio η and the components of the displacement vector are[1]:

$$u = u(x,t), \qquad v = v(x,y) = -\eta y u'_x, \qquad w = w(x,z) = -\eta z u'_x \tag{6}$$

and the equation of the motion in the compact form is given as follow:

$$\left(\frac{\partial \Lambda}{\partial \dot{u}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u'_x}\right) - \frac{\partial^2}{\partial x \partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}'_x}\right) - \frac{\partial \Lambda}{\partial u} = 0 \tag{7}$$

with the associated boundary conditions $\left.\frac{\partial \Lambda}{\partial u'_x} - \frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}'_x}\right)\right|_{x=0,l} = 0$ or $u(x,t)\big|_{x=0,l} = 0$, where

$$\Lambda = \Lambda(u, \dot{u}, u'_x, \dot{u}'_x) = \frac{1}{2}\rho(x)\left[A(x)\dot{u}^2 + \eta(x)I_p(x)\dot{u}'^2_x - A(x)E(x)u'^2_x\right] + \int_0^l A(x)F(x,t)u\,dx \tag{8}$$

is the Lagrangian density in which: I_p is the polar moment of inertia ($I_p = I_2 = \int_s r^2 ds = \pi \frac{R^4}{2}$ for the circular cross-section). By substituting (8) into (7) we obtain the Rayleigh-Love equation for the vibrating rod:

$$\underbrace{\rho(x)A(x)\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(E(x)A(x)\frac{\partial u}{\partial x}\right)}_{\text{second order wave equation}}-\underbrace{\frac{\partial}{\partial x}\left(\rho(x)\eta^2(x)I_p(x)\frac{\partial^3 u}{\partial x\partial t^2}\right)}_{\text{Rayleigh-Love correction}}+A(x)F(x,t)=0 \qquad (9)$$

The orthogonality conditions fulfilled by the eigenfunctions $X_n(x),\ n=1,2,\dots$, solution of the Strum-Liouville problem corresponding to (9) are:

$$\int_0^l \rho\left[AX_n(x)X_m(x)+\eta^2 I_p X'_n(x)X'_m(x)\right]dx=0 \ \text{ and } \int_0^l EAX'_n(x)X'_m(x)\,dx=0 \ \text{ for } \ m\neq n \qquad (10)$$

The solution of the problem is:

$$u(x,t)=\int_0^l A\left[g(\xi)\frac{\partial G_2(x,\xi,t)}{\partial t}+h(\xi)G_2(x,\xi,t)\right]d\xi+\int_0^l \eta^2 I_p\left[g'(\xi)\frac{\partial^2 G_2(x,\xi,t)}{\partial\xi\partial t}+h'(\xi)\frac{\partial G_2(x,\xi,t)}{\partial\xi}\right]d\xi+$$

$$+\frac{A}{\rho}\int_0^l\int_0^t F(\xi,\tau)G_2(x,\xi,t-\tau)\,d\tau\,d\xi \qquad (11)$$

where $G_2(x,\xi,t)=\sum_{n=1}^{\infty}\frac{X_n(x)X_n(\xi)\sin\Omega_{2,n}t}{\Omega_{2,n}\left\|X_n\right\|_1^2}$ is the Green function and $\Omega_{2,n}=\sqrt{\frac{AE}{\rho}\frac{\left\|X_n\right\|_2}{\left\|X_n\right\|_1}}$ is the eigen-values associated with the above eigenfunctions.

4. Rayleigh-Bishop theory

Certain assumptions (longitudinal and transverse displacements inside the excited rod) in the previous theory are kept and the vector displacement is the same. But in 1952, in order to improve the Rayleigh-Love theory, Bishop showed the contribution of shear stiffness accompanying the transverse deformation while calculating the strain energy[1, 2].

The general form of the equation of motion is:

$$\frac{\partial}{\partial t}\left(\frac{\partial\Lambda}{\partial\dot u}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\Lambda}{\partial u'_x}\right)-\frac{\partial^2}{\partial x\partial t}\left(\frac{\partial\Lambda}{\partial\dot u'_x}\right)-\frac{\partial^2}{\partial x^2}\left(\frac{\partial\Lambda}{\partial u''_{xx}}\right)-\frac{\partial\Lambda}{\partial u}=0 \qquad (12)$$

with the natural associated boundary conditions $\left.\frac{\partial\Lambda}{\partial u'_x}-\frac{\partial}{\partial t}\left(\frac{\partial\Lambda}{\partial\dot u'_x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial\Lambda}{\partial u''_{xx}}\right)\right|_{x=0,l}=0$, and $\left.u'_x(x,t)\right|_{x=0,l}=0$, or $\left.u(x,t)\right|_{x=0,l}=0$, and $\left.u''_{xx}(x,t)\right|_{x=0,l}=0$, where

$$\begin{aligned}\Lambda=\Lambda(u,\dot u,u'_x,\dot u'_x,u''_{xx})&=\frac{1}{2}\rho(x)\left[A(x)\dot u^2+\eta(x)I_p(x)\dot u'^2_x\right]\\&-\frac{1}{2}\left[A(x)E(x)u'^2_x+\mu(x)I_p(x)u''^2_{xx}\right]+\int_0^l A(x)F(x,t)u\,dx\end{aligned} \qquad (13)$$

is the Lagrangian density of the vibrating rod. By substituting (13) into (12), we obtain the explicit form of the equation:

$$\underbrace{\rho(x)A(x)\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(E(x)A(x)\frac{\partial u}{\partial x}\right)}_{\text{second order wave equation}}-\underbrace{\frac{\partial}{\partial x}\left(\rho(x)\eta^2(x)I_p(x)\frac{\partial^3 u}{\partial x\partial t^2}\right)}_{\text{Rayleigh correction}}+\underbrace{\frac{\partial^2}{\partial x^2}\left(\mu(x)\eta^2(x)I_p(x)\frac{\partial^2 u}{\partial x^2}\right)}_{\text{Bishop correction}}-$$

$$-A(x)F(x,t)=0 \qquad (14)$$

The investigation of the Sturm-Liouville problem corresponding to equation (14), show that the resulting eigenfunctions $X_n(x),\ n=1,2,\dots$ fulfil two orthogonality conditions:

$$\int_0^l \rho\left[AX_n(x)X_m(x)+\eta^2 I_p X_n'(x)X_m'(x)\right]dx=0 \text{ and } \int_0^l\left[EAX_n'(x)X_m'(x)+\mu\eta^2 I_p X_n''(x)X_m''(x)\right]dx=0 \tag{15}$$

The solution of equation (14) is as follows:

$$u(x,t)=\int_0^l A\left[g(\xi)\frac{\partial G_3(x,\xi,t)}{\partial t}+h(\xi)G_3(x,\xi,t)\right]d\xi+\int_0^l \eta^2 I_p\left[g'(\xi)\frac{\partial^2 G_3(x,\xi,t)}{\partial\xi\partial t}+h'(\xi)\frac{\partial G_3(x,\xi,t)}{\partial\xi}\right]d\xi+$$
$$+\frac{A}{\rho}\int_0^l\int_0^t F(\xi,\tau)G_3(x,\xi,t-\tau)d\tau d\xi \tag{16}$$

where $G_3(x,\xi,t)=\sum_{n=1}^{\infty}\frac{X_n(x)X_n(\xi)\sin\Omega_{3,n}t}{\Omega_{3,n}\left\|X_n\right\|_1^2}$ is the Green Function, in which $\Omega_n=\frac{\left\|X_n\right\|_2}{\sqrt{\rho}\left\|X_n\right\|_1}$ is the natural eigenvalues corresponding to $X_n(x),\ n=1,2,\ldots$.

5. Mindlin-Herrmann theory

Despite the fact that has improved the previous theories. It is necessary to emphasize the lack of physical clarity in interpretation of certain high-order modes, mainly independent shear and radial motion. In order to address this insufficiency Mindlin and Herrmann take into account the independent shearing deformation, radial displacement and distributed stress along the transversal direction[3]. According to these new ideas the displacements are represented by two independent functions:

$$u=u(x,t)=\Phi_0(x,t),\qquad v=v(x,r,t)=r\cdot\Phi_1(x,t) \tag{17}$$

where r is the distance between the points along the lateral direction of the rod.

The compact form of the system of equations of motion is given as follow:

$$\frac{d}{dt}\left(\frac{\partial\Lambda}{\partial\dot{\Phi}_k}\right)+\frac{d}{dx}\left(\frac{\partial\Lambda}{\partial\Phi_k'}\right)-\frac{\partial\Lambda}{\partial\Phi_k}=0,\qquad(k=0,1) \tag{18}$$

With the corresponding set of natural boundary conditions $\Phi_0(x,t)\big|_{x=0,l}=0,\ \Phi_1(x,t)\big|_{x=0,l}=0$, or $\left.\frac{\partial\Lambda}{\partial\Phi_1'}\right|_{x=0,l}=0,\ \left.\frac{\partial\Lambda}{\partial\Phi_0'}\right|_{x=0,l}=0,$, where

$$\Lambda=\Lambda\left(\dot{\Phi}_0,\dot{\Phi}_1,\Phi_0',\Phi_1',\Phi_1\right)$$
$$=\frac{\rho}{2}\left(A\dot{\Phi}_0^2(x,t)+I_2\dot{\Phi}_1^2(x,t)\right)-\frac{1}{2}\left((\lambda+2\mu)\Phi_0'^2A+4\lambda\Phi_0'\Phi_1A+4(\lambda+\mu)\Phi_1^2A+I_2\mu\Phi_1'^2\right) \tag{19}$$

is the Lagrange density of the rod, in which $\mu=\frac{E}{2(1+\eta)}$ and $\lambda=\frac{E\eta}{(1-2\eta)(1+\eta)}$, are Lame's constants. Substituting expression (19) into the system (18) leads to the explicit form of the system of equations in the operator form:

$$\begin{cases}\rho A\partial_t^2\Phi_0-(\lambda+2\mu)A\partial_x^2\Phi_0-2\lambda A\partial_x\Phi_1=AF(x,t)\\ 2\lambda A\partial_x\Phi_0+\rho I_p\partial_t^2\Phi_1-\mu I_p\partial_x^2\Phi_1+AS(\lambda+\mu)\Phi_1=0\end{cases} \tag{20}$$

The couple of eigenfunctions found by investigating the Sturm-Liouville problem associated to the system (14), hold the following orthogonality properties for $m\neq n$ $\int_0^l\left(A\Phi_{0,n}\Phi_{0,m}+I_p\Phi_{1,n}\Phi_{1,m}\right)dx=0$ and

$$\int_0^l\left\{4A(\lambda+\mu)\Phi_{1,n}\Phi_{1,m}+I_p\mu\Phi_{1,n}'\Phi_{1,m}'+A(\lambda+2\mu)\Phi_{0,n}'\Phi_{0,m}'+2\lambda A\left(\Phi_{0,n}'\Phi_{1,m}+\Phi_{0,m}'\Phi_{1,n}\right)\right\}dx=0.$$

The solution of the system of equation (20) can be found:

$$\Phi_0(x,t)=\int_0^l Ag(\xi)\frac{\partial G_4(x,\xi,t)}{\partial t}d\xi+\int_0^l I_p\phi(\xi)\frac{\partial G_5(x,\xi,t)}{\partial t}d\xi+$$
$$+\int_0^l Ah(\xi)G_4(x,\xi,t)d\xi+\int_0^l I_p\varphi(\xi)G_5(x,\xi,t)d\xi+\frac{1}{\rho}\int_0^t\int_0^l F(\xi,\tau)G_4(x,\xi,t-\tau)d\tau d\xi \quad (21)$$
$$\Phi_1(x,t)=\int_0^l Ag(\xi)\frac{\partial G_6(x,\xi,t)}{\partial t}d\xi+\int_0^l I_p\phi(\xi)\frac{\partial G_7(x,\xi,t)}{\partial t}d\xi+$$
$$+\int_0^l Ah(\xi)G_6(x,\xi,t)d\xi+\int_0^l I_p\varphi(\xi)G_7(x,\xi,t)d\xi+\frac{1}{\rho}\int_0^t\int_0^l F(\xi,\tau)G_6(x,\varepsilon,t-\tau)d\tau d\xi$$

where $u(x,r,t)\big|_{t=0}=\phi(x)$ and $\dot{u}(x,r,t)\big|_{t=0}=\varphi(x)$ are initial transverse displacement and velocity,

$$G_4(x,\xi,t)=\sum_{n=1}^{\infty}\left(\frac{\Phi_{0,n}(x)\Phi_{0,n}(\xi)\sin\Omega_{4,n}t}{\Omega_{4,n}\left\|(\Phi_{0,n},\Phi_{1,n})\right\|_1^2}\right),G_5(x,\xi,t)=\sum_{n=1}^{\infty}\left(\frac{\Phi_{0,n}(x)\Phi_{1,n}(\xi)\sin\Omega_{4,n}t}{\Omega_{4,n}\left\|(\Phi_{0,n},\Phi_{1,n})\right\|_1^2}\right),$$
$$G_6(x,\xi,t)=\sum_{n=1}^{\infty}\left(\frac{\Phi_{1,n}(x)\Phi_{0,n}(\xi)\sin\Omega_{n}t}{\Omega_{4,n}\left\|(\Phi_{0,n},\Phi_{1,n})\right\|_1^2}\right),G_7(x,\xi,t)=\sum_{n=1}^{\infty}\left(\frac{\Phi_{1,n}(x)\Phi_{1,n}(\xi)\sin\Omega_{4,n}t}{\Omega_{4,n}\left\|(\Phi_{0,n},\Phi_{1,n})\right\|_1^2}\right),$$

are the Green functions, where $\Omega_{4,n}=\dfrac{\left\|(\Phi_{0,n},\Phi_{1,n}),(\Phi_{0,n},\Phi_{1,n})'\right\|_2}{\sqrt{\rho}\left\|(\Phi_{0,n}\Phi_{1,n})\right\|_1}$ are the eigenvalues.

6. Multimode theories

A more accurate description of the rod deformation can be obtained by increasing the number of possible deformation modes. The Mindlin-McNiven theory[4] is one of possible multimode models of the rod. Here we consider another multimode generalization of the Mindlin-Herrmann model of longitudinal vibrations of the rod with circular cross-section. Assume the axisymmetric case where the displacements are approximated as follows:

$$u=u(x,r,t)=\Phi_0(x,t)+r^2\Phi_2(x,t)+...+r^{2i}\Phi_{2i}(x,t);\qquad i=0,1,2,...$$
$$v=v(x,r,t)=r\Phi_1(x,t)+r^3\Phi_3(x,t)+...+r^{2j+1}\Phi_{2j+1}(x,t),\qquad j=0,1,2,... \quad (22)$$

According to the choice of i and j we can obtain a higher or lower mode of vibration of rod.

In this case $i=1$, $j=0$: $u=u(x,r,t)=\Phi_0(x,t)+r^2\Phi_2(x,t)$ and $v=v(x,r,t)=r\Phi_1(x,t)$ and the system of equation of motion in the general form is

$$\frac{\partial}{\partial t}\left(\frac{\partial\Lambda}{\partial\dot{\Phi}_k}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\Lambda}{\partial\Phi_k'}\right)-\frac{\partial\Lambda}{\partial\Phi_k}=0,\qquad (k=0,1,2) \quad (23)$$

with a set of natural boundary conditions $\Phi_k(x,t)\big|_{x=0,l}=0$ and $\dfrac{\partial\Lambda}{\partial\Phi_k(x,t)}\Big|_{x=0,l}=0$, where

$$\Lambda=\Lambda\left(\dot{\Phi}_0,\dot{\Phi}_1,\dot{\Phi}_2,\Phi_0',\Phi_1',\Phi_2',\Phi_1,\Phi_2\right)$$
$$=\frac{1}{2}\left[\left(A\dot{\Phi}_0^2+2I_2\dot{\Phi}_0\dot{\Phi}_2+I_2\dot{\Phi}_1^2+I_4\dot{\Phi}_2^2\right)-(\lambda+2\mu)A\Phi_0'^2+\mu I_2\Phi_1'^2+(\lambda+2\mu)I_4\Phi_2'^2+4\lambda A\Phi_0'\Phi_1\right]-$$
$$-\frac{1}{2}\left[4\mu I_2\Phi_1'\Phi_2+2(\lambda+2\mu)I_2\Phi_0'\Phi_2'+4\lambda I_2\Phi_2'\Phi_1+4(\lambda+\mu)A\Phi_1^2+4\mu I_2\Phi_2^2\right] \quad (24)$$

is the Lagrangian density of the rod and $I_4=\int_s r^4 ds=\pi\dfrac{R^6}{3}$.

Substituting expression (24) into system (23) we obtain the explicit form of the system of equation of motion in the operator form:

$$A\left[\rho\partial_t^2-(\lambda+2\mu)\partial_x^2\right]\Phi_0-\left[2\lambda A\partial_x\right]\Phi_1+I_2\left[\rho\partial_t^2-(\lambda+2\mu)\partial_x^2\right]\Phi_2=0$$
$$\left[2\lambda A\partial_x\right]\Phi_0+\left[I_2\left(\rho\partial_t^2-\mu\partial_x^2\right)+4A(\lambda+\mu)\right]\Phi_1+\left[2(\lambda-\mu)I_2\partial_x\right]\Phi_2=0 \qquad (25)$$
$$I_2\left[\rho\partial_t^2-(\lambda+2\mu)\partial_x^2\right]\Phi_0-\left[2(\lambda-\mu)I_2\partial_x\right]\Phi_1+\left[I_4\left(\rho\partial_t^2-(\lambda+2\mu)\partial_x^2\right)+4\mu I_2\right]\Phi_2=0$$

The orthogonality conditions are:

$$\int_0^l\left[A\Phi_{0,n}\Phi_{0,m}+I_p\left(\Phi_{0,n}\Phi_{2,m}+\Phi_{2,n}\Phi_{0,m}+\Phi_{1,n}\Phi_{1,m}\right)+I_4\Phi_{2,n}\Phi_{2,m}\right]dx=0,$$
$$\int_0^l\left\{A\left[(\lambda+2\mu)\Phi'_{0,n}\Phi'_{0,m}+4(\lambda+\mu)\Phi_{1,n}\Phi_{1,m}+2\lambda\left(\Phi'_{0,n}\Phi_{1,m}+\Phi'_{1,n}\Phi_{0,m}\right)\right]\right.$$
$$+I_p\left[(\lambda+2\mu)\left(\Phi'_{0,n}\Phi'_{2,m}+\Phi'_{2,n}\Phi'_{0,m}\right)+2\lambda\left(\Phi_{1,n}\Phi'_{2,m}+\Phi_{1,m}\Phi'_{2,n}\right)\right.$$
$$\left.\left.+\mu\left(\Phi'_{1,n}\Phi'_{1,m}+2\left(\Phi'_{1,n}\Phi_{2,m}+\Phi'_{1,m}\Phi_{2,n}\right)+4\Phi_{2,m}\Phi_{2,n}\right)\right]+I_4(\lambda+2\mu)\Phi'_{2,m}\Phi'_{2,n}\right\}dx=0$$

for $m\neq n$.

7. Comparison of Different Models

We analyze different models of longitudinal vibrations of rods by drawing their spectral curves and compare them with the curves of the exact Pochammer-Chree solution[5, 6, 7] of the axi-symmetric problem of cylindrical rod with free outer surface. To make this comparison we assume $u(x,t)=U\cdot e^{i(\omega t-kx)}$, $\Phi_k(x,r,t)=\Phi_k(r)\cdot e^{i(\omega t-kx)}$ and substitute these values in (3), (9),(14),(20), and (25). It is supposed in this case that all parameters of equations are constant (say, $A(x)=A=const$, etc.). In the classical case we obtain a single spectral line $\omega(k)=\sqrt{E/\rho}\cdot k$. The spectral curve $\left(\omega\cdot R\cdot\sqrt{\rho/\mu}\right)$ versus of $(k\cdot R)$ of the Rayleigh-Love model is shown in Figure 1 for $k\cdot R\in(0,20]$, $\omega\cdot R\cdot\sqrt{\rho/\mu}\in(0,32]$, where R – radius of outer cylindrical surface of the rod (all other figures are drawn in the same ranges). Figure 2 demonstrates the spectral curve of the Rayleigh-Bishop model.

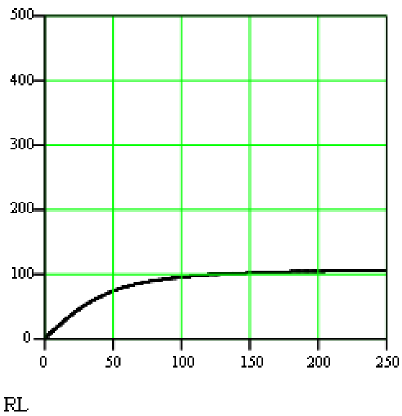

Figure 1. Rayleigh-Love model.

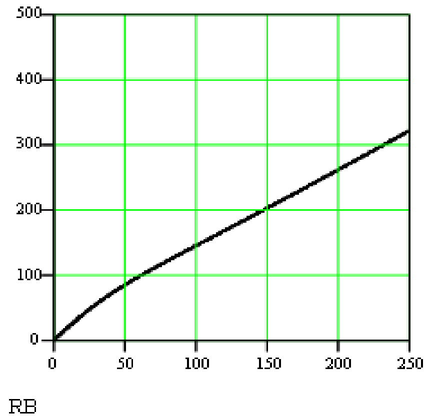

Figure 2. Rayleigh-Bishop model.

The spectral curves of the Mindlin-Herrmann model are shown in Figure 3. Figure 4 illustrates the multimode model with $i=1,\ j=0$.

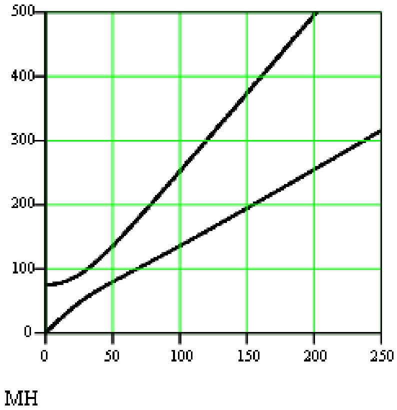

Figure 3. Mindlin-Herrmann model

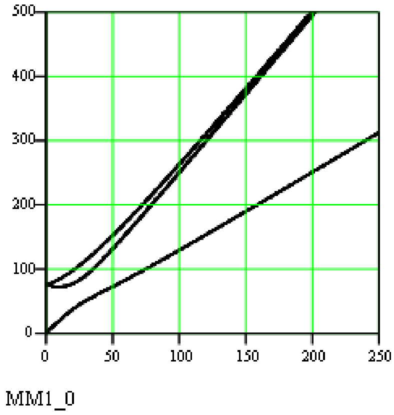

Figure 4. Multimode model ($i = 1,\ j = 0$)

Figure 5 illustrates the multimode model with $i = 2,\ j = 1$ and Figure 6 shows the exact Pochhammer–Chree model of the axisymmetric case and free cylindrical surface.

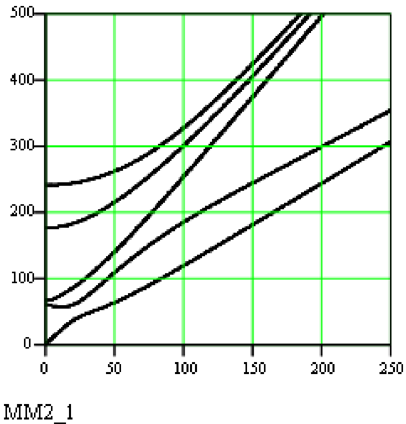

Figure 5. Multimode model ($i = 2,\ j = 1$)

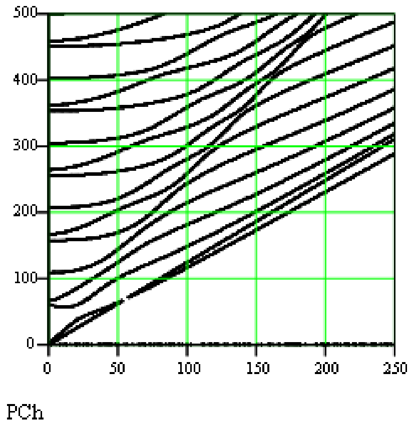

Figure 6. Pochhammer –Chree model (straight line demonstrates the shear mode)

8. Discussion and conclusions

In the present paper we compared the classical, Rayleigh-Love, Rayleigh-Bishop, Mindlin-Herrmann, and multimode models of longitudinal vibrations of rods with the exact Pochhammer-Chree solutions of axisymmetric vibration of isotropic cylinder with free surface. The classical, Rayleigh-Love, and Rayleigh-Bishop models approximately describe the first mode of the exact solution in the restricted "k - ω"- domain. The Rayleigh-Bishop approximation is more accurate, but the spectral curve asymptotically tends to the shear wave solution while the exact solution tends to the surface waves mode. It is explained by the hypothesis on plane cross-section used in the classical, Rayleigh-Love, and Rayleigh-Bishop models. The Mindlin-Herrmann model also satisfies the plane cross-section hypothesis. Due to the fact that this model is described in terms of two independent functions the set of spectral curves contain two branches. In the multimode model we reject the hypothesis on plane cross-section and obtain more spectral curves. The higher the order of the multimode approximation the broader is the "k - ω"- domain in which the effects of longitudinal vibrations of the rods could be analyzed.

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