Solution of heat equation with variable coefficient using derive

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Abstract

In this paper, the method of approximating solutions of partial differential equations with variable coefficients is studied. This is done by considering heat flow through a one-dimensional model with variable cross-sections. Two cases are considered. The first one involves quadratic approximation of the variable coefficient by direct integration. This case is studied using a conic domain. The second case approximates the variable coefficient quadratically and by step functions. The solution of the problem in each case is expressed using Green's function, and the results are compared. By the suitable use of a computer algebra system (CAS) all of these ideas can easily enough be introduced at the advanced undergraduate level.

1 INTRODUCTION

In this paper the method of approximating solutions of PDEs with variable coefficients is based on the study of the heat from equation [1], that is,

$$c\rho A(x)\frac{\partial x}{\partial t} = \frac{\partial}{\partial x}\left[kA(x)\frac{\partial u}{\partial x}\right] + A(x)f(x,t). \qquad (1)$$

In standard undergraduate courses of PDEs one considers a heat equation with constant coefficients. Equation (1) with variable coefficient is usually solved using direct numeric methods. In this paper we give an easy to implement analytical solution of the problem using the derived Green's function at a level that senior undergraduates will be able to understand and implement, using CAS.

The heat equation under study is considered with a variable cross-section area $A(x)$. In this case $A(x) = \pi(\alpha x + \beta)^2$ is the cross-sectional area of the domain and $f(x,t) = 0$. The heat capacity is c, ρ is the density of the domain and k is the thermal conductivity of the domain. The coefficients c, ρ and k are assumed to be constants for this investigation. Equation (1) is considered with boundary

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conditions of the first kind described by $u(0,t)=0$ and $u(l,t)=0$, and initial condition of the form $u(x,0)=g(x)$.

This problem describes the process of heat transfer in an axisymmetric body with cross section *A(x)* oriented so that the *x*-axis lies along the axis of the body. The case where $A(x)=\pi(\alpha x+\beta)^2 \equiv \pi\alpha^2(x+\gamma)^2$, is where $\gamma=\dfrac{\beta}{\alpha}$, corresponds to linear dependence of the boundary equation. This corresponds to the conic shape of the considered domain. It is possible to find an exact solution expressed by Green's function.

The linear function $\alpha x+\beta$ can be approximated by a step function. Physically this corresponds to a cylinder consisting of *N* sections of constant cross sections $A_j\,(j=1,2,...,N)$. The case for $N>2$ was considered by Fedotov, *et al* (see [2] and [3]). The Green's function can be obtained subject to the solution satisfying the boundary conditions at the junctions. The continuity of the solution at the junctions is described as follows:

$$u_j(l_j,t)=u_{j+1}(l_j,t),$$

and the continuity heat flow is given by:

$$A_j u'(l_j,t)=A_{j+1}u'(l_{j+1},t), \qquad (j=1,2\cdots N-1)\,.$$

The initial condition in our example is defined by:

$$g(x)=x(4-x)\,.$$

The results can be considered as an approximation of the solution for variable cross section $A(x)$, see Figure 2. The results of both methods will be compared and the solutions at the end of the paper show approximate similarity of results.

2 Model of a one-dimensional domain with variable cross-sections governing the PDE and boundary conditions

2.1 Approximating by linear functions

The model of a conic domain of variable cross-sections of *N* sections is illustrated by Figure 1 below.

The solution to Equation (1) is sought in the form:

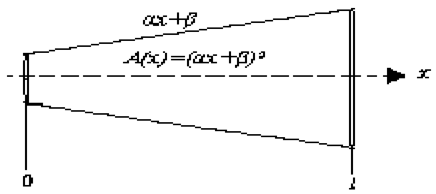
$$u(x,t)=\sum_{n=1}^{\infty} b_n X_n(x) e^{-\lambda_n^2 t},$$

where:

$$X_n(x) = \frac{\sin \lambda_n x}{(x+\gamma)}, \qquad (n = 1, 2, \ldots,) \tag{2}$$

Figure 1

is obtained by applying the method of separation of variables to equation (1) with application of the boundary conditions of the first kind.



It is clear that the System (2) is orthogonal with weight $(x+\gamma)^2$. It follows from equation (2) that:

$$\lambda_n = \frac{n\pi}{l}$$

and this formula gives the eigenvalues. An example is considered where *l* = *4* and γ = *1*. The eigenvalues obtained for this example are shown in section 3. The solution to the problem is given by the following equation:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{\varphi_n^2} \int_0^l A(\xi) X_n(x) X_n(\xi) g(\xi) e^{-\lambda_n^2 t} d\xi \tag{3}$$

where

$$\varphi_n^2 = \int_0^l A(\xi) X_n^2(\xi) d\xi = \int_0^l (x+\gamma)^2 \frac{\sin^2 \lambda_n x}{(x+\gamma)^2} dx = \frac{l}{2}, \qquad (n = 1, 2, \ldots,)$$

is the norm squared, and

$$g(x) = \sum_{n=1}^{\infty} b_n \left[\frac{\sin \frac{n\pi}{l} x}{(x+\gamma)} \right]$$

is the initial function. The coefficients b_n are described as follows:

$$b_n = \frac{2}{l} \int_0^l A(\xi) X_n(\xi) g(\xi) d\xi, \qquad (n = 1, 2, \ldots,).$$

The solution given by equation (3) can also be represented by

$$u(x,t)=\int_0^l G(x,\xi,t)g(\xi)d\xi\ ,$$

where

$$G(x,\xi,t)=\sum_{n=1}^{\infty}\frac{2}{l}A(\xi)X_n(x)X_n(\xi)e^{-\lambda_n^2 t}$$

is Green's function.

2.2 Approximating by using step functions

The domain for this case is illustrated in Figure 2 below.

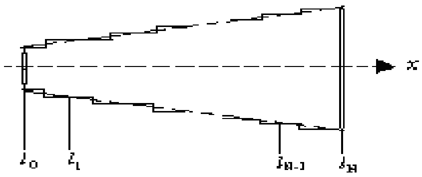


Figure 2

The solution is also sought in the form:

$$u(x,t)=\sum_{m=1}^{\infty}b_m X_m(x)e^{-\lambda_n^2 t},\qquad (n=1,2,...,)\ ,$$

where in this case:

$$X_m(x)=\sum_{j=1}^{N}X_j^{(m)}\theta_j(x),$$

$$\theta_j(x)=H(x-l_{j-1})-H(x-l_j),\qquad (j=1,2,...,)$$

is the Heaviside function difference, and:

$$H(x)=\begin{cases}0 & x<0\\ 1 & x>0.\end{cases}$$

The j^{th} eigenfunction is expressed as follows:

$$X_j^{(m)}(x)=c_{2j-1}\cos\lambda x+c_{2j}\sin\lambda x, \tag{4}$$

and is obtained as described under the conic domain case, with:

$$b_m=\frac{1}{\varphi_m^2}\int_0^{l_N}X_m(\xi)g(\xi)d(\xi)\ ,$$

and *g(x)* is the initial condition described as follows:

$$g(x)=\sum_{m=1}^{\infty} b_m X_m(x).$$

Application of the boundary conditions at the endpoints and at the junctions to equation (4), results in an (*N*×*N*) block matrix, which is helpful in determining the eigenvalues. The solution of the problem is given by the following equation:

$$u(x,t)=\sum_{m=1}^{\infty}\frac{1}{\varphi_m^2}\int_0^{l_N} A(\xi)X_m(x)X_m(\xi)g(\xi)e^{-\lambda_n^2 t}d(\xi), \tag{5}$$

where the weight function is:

$$A(x)=\begin{cases} A_1 & l_0\le x\le l_1\\ A_2 & l_1\le x\le l_2\\ \vdots & \vdots\\ A_{n-1} & l_{N-2}\le x\le l_{N-1}\\ A_N & l_{N-1}\le x\le l_N\end{cases},$$

and the norm squared is described as follows:

$$\varphi_m^2=\int_0^{l_N} A(\xi)X_m^2(\xi)d\xi.$$

The solution is also given in the form:

$$u(x,t)=\int_0^{l_N} G(x,\xi,t)g(\xi)d\xi\ ,$$

where:

$$G(x,\xi,t)=\sum_{m=1}^{\infty}\frac{1}{\varphi_m^2}A(\xi)X_m(x)X_m(\xi)e^{-\lambda_m^2 t}$$

is Green's function.

3 Results

3.1 The conic domain case

The first five eigenvalues of the smooth functions in the conic case, as an example, where $l=4$, are given in the Table 1.

Table 1: Eigenvalues

λ_1	λ_2	λ_3	λ_4	λ_5
$\frac{\pi}{4}=0.78540$	$\frac{\pi}{2}=1.5708$	$\frac{3\pi}{4}=2.3562$	$\pi=3.1416$	$\frac{5\pi}{4}=3.9270$

The first five eigenfunctions corresponding to the eigenvalues in Table 1 are shown in Table 2.

Table 2: Eigenfunctions

$X_1(x) = \frac{\sin \frac{\pi}{4} x}{(x+1)}$
$X_2(x) = \frac{\sin \frac{\pi}{2} x}{(x+1)}$
$X_3(x) = \frac{\sin \frac{3\pi}{4} x}{(x+1)}$
$X_4(x) = \frac{\sin \pi x}{(x+1)}$
$X_5(x) = \frac{\sin \frac{5\pi}{4} x}{(x+1)}$

Graphs of the smooth eigenfunctions in Table 2 are represented in f

Figure 3, where derive was used.

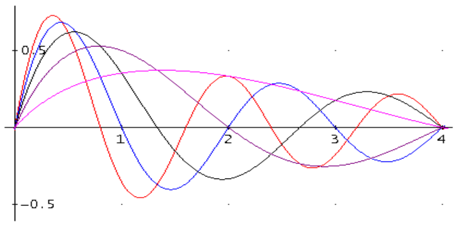


Figure 3

Figure 4, which was obtained using derive, represents the analytical solution of the problem.

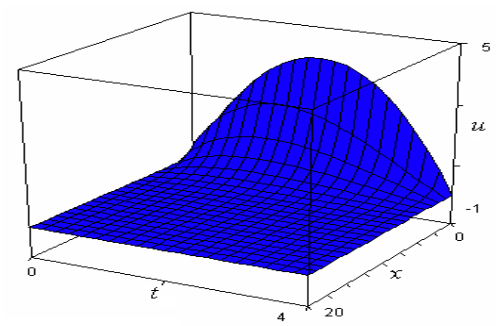


Figure 4

3.2 The stepped domain case

To obtain the eigenvalues for the stepped case, an example with $l_N = 4$ is also considered. For this example, an (8×8) matrix was obtained from the following system of equations.

$$\left.\begin{aligned}
&c_1 \cos\lambda.0 + c_2 \sin\lambda.0 = 0 \Rightarrow c_1 = 0\\
&c_1 \cos\lambda.l_1 + c_2 \sin\lambda.l_1 - c_3 \cos\lambda.l_1 - c_4 \sin\lambda.l_1 = 0\\
&A_1\lambda[-c_1 \sin\lambda.l_1 + c_2 \cos\lambda.l_1] - A_2\lambda[-c_3 \sin\lambda.l_1 + c_4 \cos\lambda.l_1] = 0\\
&c_3 \cos\lambda.l_2 + c_4 \sin\lambda.l_2 - c_5 \cos\lambda.l_2 - c_6 \sin\lambda.l_2 = 0\\
&A_2\lambda[-c_3 \sin\lambda.l_2 + c_4 \cos\lambda.l_2] - A_3\lambda[-c_5 \sin\lambda.l_2 + c_6 \cos\lambda.l_2] = 0\\
&c_5 \cos\lambda.l_3 + c_6 \sin\lambda.l_3 - c_7 \cos\lambda.l_3 - c_8 \sin\lambda.l_3 = 0\\
&A_3\lambda[-c_5 \sin\lambda.l_3 + c_6 \cos\lambda.l_3] - A_4\lambda[-c_7 \sin\lambda.l_3 + c_8 \cos\lambda.l_3] = 0\\
&c_7 \cos\lambda.l_4 + c_8 \sin\lambda.l_4 = 0
\end{aligned}\right\} \qquad (6)$$

The variable area for the domain with variable cross sections of four sections is described as follows:

$$A(x) = \begin{cases} A_1 = \left(\frac{1}{2}+1\right)^2 = 2.25 & l_0 \le x \le l_1 \\ A_2 = \left(\frac{3}{2}+1\right)^2 = 6.25 & l_1 \le x \le l_2 \\ A_3 = \left(\frac{5}{2}+1\right)^2 = 12.25 & l_2 \le x \le l_3 \\ A_4 = \left(\frac{7}{2}+1\right)^2 = 20.25 & l_3 \le x \le l_4 \end{cases}.$$

The following table shows the first five eigenvalues obtained from solution of the transcedental System (6), using the method of finding roots as used by Fedotov, *et al* [5].

Table 3: Eigenvalues

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}
0.7874	1.5708	2.3542	3.14159	3.929	4.71239	5.4958	6.28319	7.0706	7.85398

The continuous eigenfunctions, with discontinuous derivatives due to approximation, using non-smooth functions and corresponding to the eigenvalues in Table 3, are given in Table 4.

Table 4: Eigenfunctions

$X_1(x) = c_1 \cos(0.7874x) + c_2 \sin(0.7874x)$
$X_2(x) = c_3 \cos(1.5708x) + c_4 \sin(1.5708x)$
$X_3(x) = c_5 \cos(2.3542x) + c_6 \sin(2.3542x)$
$X_4(x) = c_7 \cos(3.14159x) + c_8 \sin(3.14159x)$.

Table 5 shows the values of the coefficients $c_1, c_2, \ldots, c_8$.

Table 5: Coefficients values

λ	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
λ_1	0	1	0.32	0.68128	0.16193	0.68065	−0.00411	0.5126
λ_2	0	1	0	0.5102	0	0.5102	0	0.5102
λ_3	0	1	−0.32	0.68128	−0.16193	0.68065	0.00411	0.5126
λ_4	0	1	0	0.36	0	0.18367	0	0.1111
λ_5	0	1	0.32	0.68128	0.16193	0.68065	−0.00411	0.5126

Mathcad was used to obtain graphs of the eigenfunctions in Table 4, as depicted in Figure 5.

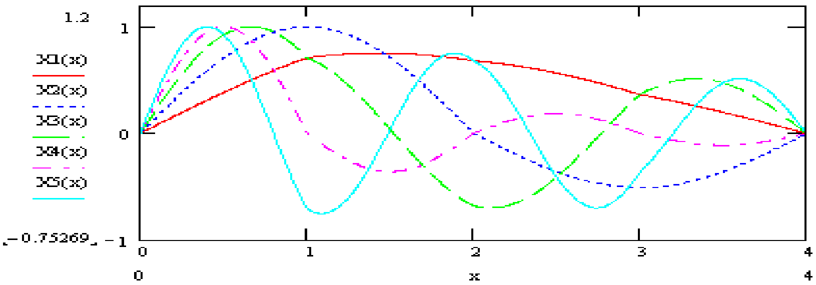


Figure 5

Figure 6 gives the solution of the problem (USING Mathcad). Notice again the jump of the derivatives at the points where junctions occur. This is a property of eigenfunctions.

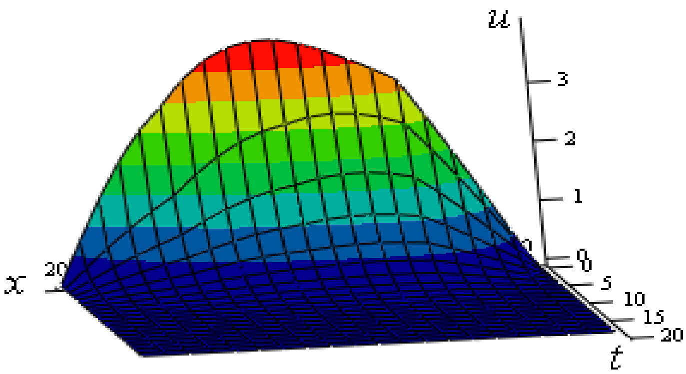


Figure 6

4 Conclusion

In this paper approximations of solutions of partial differential equations in a domain of varying area were studied, using CAS and techniques that are easy enough for senior undergraduate students to understand. Two cases were considered. The first one involved a conic domain where the equation under study was approximated by smooth eigenfunctions. The second case involved a stepped domain and here the equation was approximated by non-smooth eigenfunctions. The use of technology such as derive and Matlab yielded similar analytical solutions, as shown in Figures 4 and 6 (which are closely related).This shows that the methods of approximating solutions of PDEs give similar results. Different kinds of boundary conditions may be considered, for example, boundary conditions of the second and third kinds [4]. In this paper these boundary conditions were not presented for the sake of simplicity. This concept can be extended to domains of more complicated shapes. An example of such a shape that can be approximated either by linear or step functions is illustrated by Figures 7(a) and 7(b).

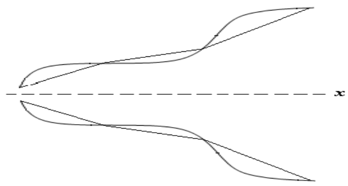


Figure 7(a)

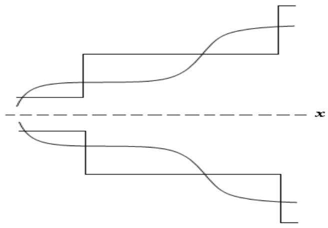


Figure 7(b)

These figures represent the domain of an arbitrary shape approximated first by using linear functions (Figure 7(a)) and second by step functions (Figure 7(b)).

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