A THEORY OF LONGITUDINAL VIBRATIONS OF AN ISOTROPIC BAR BASED ON THE MINDLIN-HERRMANN MODEL

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Abstract

Modern theories of one-dimensional bar vibrations account for lateral effects, which are substantial in the case of relatively thick bars. For example, in the Rayleigh-Love and Rayleigh-Bishop models the lateral displacements are supposed to be proportional to the product of longitudinal strain of the bar, its Poisson ratio and the distance from the neutral line of the cross-section. In the Mindlin-Herrmann model the lateral displacements are independent of longitudinal stress and Poisson ratio and proportional to the product of a new dependent function and the distance from the neutral line of the bar. Hamilton's variational principle is used for correct formulation of the boundary conditions. In this approach a system of equations and possible boundary conditions are obtained. In this case the mathematical model of the bar is described by a system of two partial differential equations of second order, which could be transformed to a single partial differential equation of the fourth order. It is shown how a new Lagrangian may be calculated so as to directly obtain the fourth order equation of the model by application of the Hamilton variational principle. Another major advantage of the variational approach is in the natural formulation of orthogonality conditions for eigenfunctions. Two orthogonal conditions are proven and used to derivation the Green's function in which the general solution of the problem is formulated. The main theoretical results of the paper are as follows: formulation and proof of two types of orthogonality conditions, presentation of a new Lagrangian in terms of the conventional strain and kinetic energy as well as an energy of accelerations of the bar, and derivation of the general solution in terms of the Green's function.

I **Mathematical model**

Displacements according to Mindlin and Herrmann[ ] are assumed as follows:

$$u = u(x,t); \quad v = v(x,y,t) = y \cdot \psi(x,t); \quad w = w(x,z,t) = z \cdot \psi(x,t) \tag{1}$$

Where x is the axial distance along the bar, y is the lateral distance from the neutral line t is the time, u is the axial displacement, ψ is transverse contraction and v, w are transverse or lateral displacements

Strains (geometrical characteristic of the deformation):

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = u'; \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = \psi; \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} = \psi = \varepsilon_{yy};$$
$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = y\psi'; \quad \varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0; \quad \varepsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = z\psi' \tag{2}$$

Stresses (physical characteristic of the deformation):

$$\begin{aligned}
\sigma_{xx} &= (\lambda + 2\mu)\varepsilon_{xx} + \lambda(\varepsilon_{yy} + \varepsilon_{zz}) = (\lambda + 2\mu)u' + 2\lambda\psi; \\
\sigma_{yy} &= (\lambda + 2\mu)\varepsilon_{yy} + \lambda(\varepsilon_{xx} + \varepsilon_{zz}) = 2(\lambda + \mu)\psi + \lambda u'; \\
\sigma_{zz} &= (\lambda + 2\mu)\varepsilon_{zz} + \lambda(\varepsilon_{xx} + \varepsilon_{yy}) = 2(\lambda + \mu)\psi + \lambda u' = \sigma_{yy}; \\
\sigma_{xy} &= \mu\varepsilon_{xy} = \mu y\psi'; \quad \sigma_{yz} = \mu\varepsilon_{yz} = 0; \quad \sigma_{zx} = \mu\varepsilon_{zx} = \mu z\psi'
\end{aligned} \tag{3}$$

Where λ and μ are Lame's constants defined by $\lambda = \dfrac{E\eta}{(1-2\eta)(1+\eta)}$ and $\mu = \dfrac{E}{2(1+\eta)}$ (E is the Young modulus of elasticity and η is the Poisson ratio).

Kinetic energy:

$$K = \frac{\rho}{2}\int_0^l \int_{(A)} \left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dA\, dx \tag{4}$$

Substitute (1) into (4) leads to

$$K = \frac{\rho}{2}\int_0^l \int_{(A)} \left(\dot{u}^2 + y^2\dot{\psi}^2 + z^2\dot{\psi}^2\right) dA\, dx, \text{ or}$$
$$K = \frac{\rho}{2}\int_0^l \left(A\dot{u}^2 + \dot{\psi}^2 I_p\right) dx \tag{5}$$

Where $A = \int_{(A)} dA$ is the area of the cross sectional of the bar, $I_p = \int_{(A)} \left(y^2 + z^2\right) dA$ denote the polar moment of inertia of the cross section, ρ is the mass density, l is the bar length and above and in what follows the arguments in all function are omitted for brevity.

Strain energy:

$$P = \frac{\rho}{2}\int_0^l \int_{(A)} \left(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \sigma_{xy}\varepsilon_{xy} + \sigma_{yz}\varepsilon_{yz} + \sigma_{zx}\varepsilon_{zx}\right) dA\, dx \tag{6}$$

Where $\sigma_{ij}, \varepsilon_{ij}$ are given by expressions (2) and (3).

Hence

$$P = \frac{1}{2}\int_0^l \left\{ A\left[(\lambda+2\mu)\dot{u}^2 + 4\lambda\psi u' + 4(\lambda+\mu)\psi^2 \right] + \mu I_p (\psi')^2 \right\} dx \tag{7}$$

Work of distributed force $f = f(x,t)$:

$$W = \int_0^l \int_{(A)} f \cdot u \; dA\, dx \text{ or}$$
$$W = \int_0^l f.uAdx \tag{8}$$

Lagrangian:

$$L = K - P + W = \frac{1}{2}\int_0^l \left\{ \rho A\dot{u}^2 + \rho I_p \dot{\psi}^2 - (\lambda+2\mu) Au'^2 - 4\lambda Au'\psi \right.$$
$$\left. -4(\lambda+\mu) A\psi^2 - \mu I_p \psi'^2 + 2\, fAu \right\} dx \tag{9}$$

Applying the Hamiltonian principle to the Lagrange functional (9) we obtain the system of equations of motion in general form:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{u}}\right) + \dfrac{d}{dx}\left(\dfrac{\partial L}{\partial u'}\right) - \dfrac{\partial L}{\partial u} = 0 \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\psi}}\right) + \dfrac{d}{dx}\left(\dfrac{\partial L}{\partial \psi'}\right) - \dfrac{\partial L}{\partial \psi} = 0 \end{cases} \tag{10}$$

And the associated boundary conditions in general form:

- for fixed ends:

$$[u]_{x=0,l} = 0 \quad \text{and} \quad \left[\frac{\partial L}{\partial \psi'}\right]_{x=0,l} = 0 \tag{11}$$

- for free ends:

$$[\psi]_{x=0,l} = 0 \quad \text{and} \quad \left[\frac{\partial L}{\partial u'}\right]_{x=0,l} = 0 \tag{12}$$

Equations (8) in the explicit form:

$$\begin{cases} \rho\ddot{u} - (\lambda+2\mu)u'' - 2\lambda\psi' = f(x,t) \\ \rho I_p \ddot{\psi} + 4(\lambda+\mu) A\psi - \mu I_p \psi'' + 2\lambda Au' = 0 \end{cases} \tag{10a}$$

Boundary conditions in the explicit form:

- for fixed ends:

$$[u]_{x=0,l} = 0 \quad \text{and} \quad [\psi']_{x=0,l} = 0 \tag{11a}$$

- for free ends:

$$[\psi]_{x=0,l} = 0 \quad \text{and} \quad \left[(\lambda+2\mu)u' + 2\lambda\psi\right]_{x=0,l} = 0 \tag{12a}$$

The problem is entirely described by adding the following initial conditions

$$u\big|_{t=0} = g(x),\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = h(x), \psi\big|_{t=0} = \phi(x), \left.\frac{\partial \psi}{\partial t}\right|_{t=0} = \varphi(x) \tag{13}$$

II Free vibrations of Mindlin-Herrmann bar and natural frequencies

In order to obtain the Sturm-Liouville problem for the determination of eigenvalues, we consider the harmonic vibration of the bar, which correspond to

$$f(x,t)=0,\psi(x,t)=\Psi(x,t)e^{i\omega t} \text{ and } u(x,t)=X(x)e^{i\omega t},\left(i^2=-1\right) \tag{14}$$

Substituting equations (14) into (10a) and (12a) leads to an atypical Sturm Liouville problem:

$$\begin{cases}(\lambda+2\mu)X''+2\lambda\Psi'=-\omega^2\rho I_p X\\ \mu I_p\Psi''-2A\lambda X'-4A(\lambda+\mu)\Psi=-\omega^2\rho I_p\Psi\end{cases} \tag{15}$$

Boundary conditions $\left.(\lambda+2\mu)X'+2\lambda\Psi\right|_{x=0,l}=0,$ and $\left.\Psi\right|_{x=0,l}=0$ (16)

Despite the fact that there is a lack of theory, until at the present moment to solve analytically the eigenvalues problem (15)-(16), one can use mathematical softwares to find the solution.

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III Orthogonality of the eigenfunctions

Let (X_n,Ψ_n) and $(\mathrm{X}_{\mathrm{m}},\Psi_m)$ be two distinct couple of eigenfunctions corresponding respectively to different eigenvalues ω_n and ω_{m}, satisfying the system of differential equation (15)

$$\begin{cases}(\lambda+2\mu)X_n''+2\lambda\Psi_n'=-\omega_n^2\rho I_p X_n\\ \mu I_p\Psi_n''-2A\lambda X_n'-4A(\lambda+\mu)\Psi_n=-\omega_n^2\rho I_p\Psi_n\end{cases} \tag{17}$$

$$\begin{cases}(\lambda+2\mu)X_m''+2\lambda\Psi_m'=-\omega_m^2\rho I_p X_m\\ \mu I_p\Psi_m''-2A\lambda X_m'-4A(\lambda+\mu)\Psi_m=-\omega_m^2\rho I_p\Psi_m\end{cases} \tag{18}$$

and the boundary conditions (16).

Multiply the first equation of the system (17) by $\dfrac{X_m}{\rho^2 I_p}$ and the second by $\dfrac{\Psi_m}{\rho^2 I_p}$, also multiply the first equation of the system (18) by $-\dfrac{X_n}{\rho^2 I_p}$ and the second by $-\dfrac{\Psi_n}{\rho^2 I_p}$.

Afterward add all the equations and integrate the resulting equation over x from 0 to l. Applying the technique of integration by part to terms, using boundary conditions (16) associated to the systems (17) and (18) and diving by $\left(\omega_m^2-\omega_n^2\right)$.Finally we obtain the first orthogonality of these eigenfunctions:

$$\int_0^l\left(AX_nX_m+I_p\Psi_n\Psi_m\right)dx=0,\ m\neq n \tag{19}$$

With respect to the generalized weight constant functions A and I_p. The corresponding square norm (cross section norm) is:

$$\left\|(X_n,\Psi_n)\right\|_{A,I_p}^2=\int_0^l\left(AX_n^2+I_p\Psi_n^2\right)dx \tag{20}$$

To seek the second orthogonality, we follow the previous technique for the first orthogonality but by replacing ω_n^2 and ω_m^2 respectively by $\frac{1}{k_n^2}$ and $\frac{1}{k_m^2}$ into the system (17) and (18), thus we obtain (21)

$$\int_0^l \left\{4A(\lambda+\mu)\Psi_n\Psi_m + I_p\mu\Psi_n'\Psi_m' + A(\lambda+2\mu)X_n'X_m' + 2\lambda A(X_n'\Psi_m + X_m'\Psi_n)\right\}dx = 0, m \neq n$$

which is the second orthogonality with respect of the generalized constant weight functions, $4A(\lambda+\mu), A(\lambda+2\mu), I_p\mu$ and $2\lambda A$ and the associated square norm is (22)

$$\left\|(X_n,\Psi_n),(X_n,\Psi_n)'\right\|_{\lambda,\mu}^2 = \int_0^l \left\{4A(\lambda+\mu)\Psi_n^2 + I_p\mu(\Psi_n')^2 + A(\lambda+2\mu)(X_n')^2 + 4\lambda AX_n'\Psi_n\right\}dx$$

Remark: It is possible to obtain the same or/and other kind of orthogonalities, by using any combination of two of the four boundary conditions (11a)-(12a)

IV Solution of the problem: Green function

Assume that the solution of the inhomogeneous system of the initial boundary problem (10a), (12a)-(13) can be written as a Fourier series expansion with respect of the eigenfunction system $\left\{(X_n,\Psi_n)\right\}_{n=1}^{\infty}$,

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)\Phi_n(t) \text{ and } \psi(x,t) = \sum_{n=1}^{\infty}\Psi_n(x)\Phi_n(t) \tag{23}$$

where the unknown time functions $\Phi_n(t)$ need to be determined.

Substituting expressions (23) into the Lagrange functional (9) gives

$$\begin{aligned} L = {} & \frac{1}{2}\sum_{n=1}^{\infty}\dot{\Phi}_n^2\rho\int_0^l\left\{AX_n^2 + I_p\Psi_n^2\right\}dx + \sum_{n<m}^{\infty}\dot{\Phi}_n\dot{\Phi}_m\rho\int_0^l\left\{AX_nX_m + I_p\Psi_n\Psi_m\right\}dx - \\ & -\frac{1}{2}\sum_{n=1}^{\infty}\Phi_n^2\int_0^l\left\{4A(\lambda+\mu)\Psi_n^2 + I_p\mu(\Psi_n')^2 + A(\lambda+2\mu)(X_n')^2 + 4\lambda AX_n'\Psi_n\right\}dx - \\ & -\sum_{n<m}^{\infty}\Phi_n\Phi_m\int_0^l\left\{4A(\lambda+\mu)\Psi_n\Psi_m + I_p\mu\Psi_n'\Psi_m' + A(\lambda+2\mu)X_n'X_m'\right\}dx + \\ & -\sum_{n<m}^{\infty}\Phi_n\Phi_m 2\lambda A(X_n'\Psi_m + X_m'\Psi_n) + \sum_{n=1}^{\infty}A\Phi_n\int_0^l f(x,t)X_n dx \end{aligned} \tag{24}$$

Using orthogonality conditions (19), (21) and norm formula (20), (22) in (24) gives

$$L = \frac{1}{2}\sum_{n=1}^{\infty}\left\{\rho\dot{\Phi}_n^2\left\|(X_n\Psi_n)\right\|_{A,I_p}^2 - \Phi_n^2\left\|(X_n,\Psi_n),(X_n,\Psi_n)'\right\|_{\lambda,\mu}^2 + \Phi_nA\int_0^l f(x,t)X_n dx\right\} \tag{25}$$

From the variational principle the Lagrangian (9) consequently (25) satisfy the following system of Euler-Lagrange differential equations

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\dot{\Phi}_n}\right) + \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial\Phi_n}\right) - \frac{\partial L}{\partial\Phi_n} = 0 \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\dot{\Phi}_n}\right) + \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial\Phi_n}\right) - \frac{\partial L}{\partial\Phi_n} = 0 \end{cases}, \text{ for } n = 1,2... \tag{26}$$

hence, we obtain the following ordinary differential equation with respect of time

$$\ddot{\Phi}_n(t) + \Omega_n^2\Phi_n(t) = f_n(t), \text{ for } n = 1,2... \tag{27}$$

Where $\Omega_n^2 = \dfrac{\left\|(X_n,\Psi_n),(X_n,\Psi_n)'\right\|_{\lambda,\mu}^2}{\rho\left\|(X_n\Psi_n)\right\|_{A,I_p}^2}$ and $f_n(t) = \dfrac{A}{\rho\left\|(X_n\Psi_n)\right\|_{A,I_p}^2}\int_0^l f(x,t)X_n dx$

The general solution of (27) is on the form

$$\Phi_n(t) = \Phi_n(0)\cos(\Omega_n t) + \frac{\dot{\Phi}_n(0)}{\Omega_n}\sin(\Omega_n t) + \frac{1}{\Omega_n}\int_0^t f_n(\tau)\sin[\Omega_n(t-\tau)]d\tau \qquad (28)$$

To determine conveniently constants $\Phi_n(0)$ and $\dot{\Phi}_n(0)$ we need the initial conditions (13) which should also be expanded into Fourier series with respect to eigenfunctions system

$$\begin{aligned} &u(x,0) = g(x) = \sum_{n=1}^{\infty}\Phi_n(0)X_n(x), \dot{u}(x,0) = h(x) = \sum_{n=1}^{\infty}\dot{\Phi}_n(0)X_n(x), \\ &\phi(x) = \psi(x,0) = \sum_{n=1}^{\infty}\Phi_n(0)\Psi_n(x), \varphi(x) = \psi(x,0) = \sum_{n=1}^{\infty}\dot{\Phi}_n(0)\Psi_n(x) \end{aligned} \qquad (29)$$

Using the properties of the previous expansion, orthogonality condition (19) and the norm formula (20), we can express $\Phi_n(0)$ and $\dot{\Phi}_n(0)$, as Fourier coefficients:

$$\begin{aligned} &\Phi_n(0) = \frac{1}{\left\|(X_n,\Psi_n)\right\|_{A,I_p}}\int_0^l \left(AX_n(x)g(x) + I_p\Psi_n(x)\phi(x)\right)dx \\ &\dot{\Phi}_n(0) = \frac{1}{\left\|(X_n,\Psi_n)\right\|_{A,I_p}}\int_0^l \left(AX_n(x)h(x) + I_p\Psi_n(x)\varphi(x)\right)dx \end{aligned} \qquad (30)$$

Substituting (30) into (28) and get the solution of the initial problem, afterward we substitute equation (28) in the expression (23) to obtain the solution of the problem for the longitudinal vibrations of the Mindlin-Herrmann isotropic bar

$$\begin{aligned} u(x,t) = &\int_0^l Ag(\varepsilon)\sum_{n=1}^{\infty}\left(\frac{X_n(x)X_n(\varepsilon)}{\Omega_n\left\|(X_n,\Psi_n)\right\|_{A,I_p}}\frac{\partial\sin\Omega_n t}{\partial t}\right)d\varepsilon + \int_0^l I_p\phi(\varepsilon)\sum_{n=1}^{\infty}\left(\frac{X_n(x)\Psi_n(\varepsilon)}{\Omega_n\left\|(X_n,\Psi_n)\right\|_{A,I_p}}\frac{\partial\sin\Omega_n t}{\partial t}\right)d\varepsilon + \\ &+\int_0^l Ah(\varepsilon)\sum_{n=1}^{\infty}\left(\frac{X_n(x)X_n(\varepsilon)\sin\Omega_n t}{\Omega_n\left\|(X_n,\Psi_n)\right\|_{A,I_p}}\right)d\varepsilon + \int_0^l I_p\varphi(\varepsilon)\sum_{n=1}^{\infty}\left(\frac{X_n(x)\Psi_n(\varepsilon)\sin\Omega_n t}{\Omega_n\left\|(X_n,\Psi_n)\right\|_{A,I_p}}\right)d\varepsilon + \\ &+\frac{1}{\rho}\int_0^t\int_0^l f(x,t)\sum_{n=1}^{\infty}\left(\frac{X_n(x)X_n(\varepsilon)\sin\Omega_n(t-\tau)}{\Omega_n\left\|(X_n,\Psi_n)\right\|_{A,I_p}}\right)d\tau d\varepsilon \end{aligned}$$

$$\begin{aligned} \psi(x,t) = &\int_0^l Ag(\varepsilon)\sum_{n=1}^{\infty}\left(\frac{\Psi_n(x)X_n(\varepsilon)}{\Omega_n\left\|(X_n,\Psi_n)\right\|_{A,I_p}}\frac{\partial\sin\Omega_n t}{\partial t}\right)d\varepsilon + \int_0^l I_p\phi(\varepsilon)\sum_{n=1}^{\infty}\left(\frac{\Psi_n(x)\Psi_n(\varepsilon)}{\Omega_n\left\|(X_n,\Psi_n)\right\|_{A,I_p}}\frac{\partial\sin\Omega_n t}{\partial t}\right)d\varepsilon + \\ &+\int_0^l Ah(\varepsilon)\sum_{n=1}^{\infty}\left(\frac{\Psi_n(x)X_n(\varepsilon)\sin\Omega_n t}{\Omega_n\left\|(X_n,\Psi_n)\right\|_{A,I_p}}\right)d\varepsilon + \int_0^l I_p\varphi(\varepsilon)\sum_{n=1}^{\infty}\left(\frac{\Psi_n(x)\Psi_n(\varepsilon)\sin\Omega_n t}{\Omega_n\left\|(X_n,\Psi_n)\right\|_{A,I_p}}\right)d\varepsilon + \\ &+\frac{1}{\rho}\int_0^t\int_0^l f(x,t)\sum_{n=1}^{\infty}\left(\frac{X_n(\varepsilon)\Psi_n(x)\sin\Omega_n(t-\tau)}{\Omega_n\left\|(X_n,\Psi_n)\right\|_{A,I_p}}\right)d\tau d\varepsilon \end{aligned}$$

We can introduce the Green's function in order to simplify the representation of the solution

$$u\left(x,t\right)=\int_0^l Ag(\varepsilon)\frac{\partial G_1(x,\varepsilon,t)}{\partial t}d\varepsilon+\int_0^l I_p\phi(\varepsilon)\frac{\partial G_2(x,\varepsilon,t)}{\partial t}d\varepsilon+$$

$$+\int_0^l Ah(\varepsilon)G_1(x,\varepsilon,t)d\varepsilon+\int_0^l I_p\varphi(\varepsilon)G_2(x,\varepsilon,t)d\varepsilon+\frac{1}{\rho}\int_0^t\int_0^l f(x,t)G_1(x,\varepsilon,t-\tau)d\tau d\varepsilon$$

$$\psi(x,t)=\int_0^l Ag(\varepsilon)\frac{\partial G_3(x,\varepsilon,t)}{\partial t}d\varepsilon+\int_0^l I_p\phi(\varepsilon)\frac{\partial G_4(x,\varepsilon,t)}{\partial t}d\varepsilon+$$

$$+\int_0^l Ah(\varepsilon)G_3(x,\varepsilon,t)d\varepsilon+\int_0^l I_p\varphi(\varepsilon)G_4(x,\varepsilon,t)d\varepsilon+\frac{1}{\rho}\int_0^t\int_0^l f(x,t)G_3(x,\varepsilon,t-\tau)d\tau d\varepsilon$$

Where

$$G_1(x,\varepsilon,t)=\sum_{n=1}^{\infty}\left(\frac{X_n(x)X_n(\varepsilon)\sin\Omega_n t}{\Omega_n\left\|\left(X_n,\Psi_n\right)\right\|_{A,I_p}}\right),G_2(x,\varepsilon,t)=\sum_{n=1}^{\infty}\left(\frac{X_n(x)\Psi_n(\varepsilon)\sin\Omega_n t}{\Omega_n\left\|\left(X_n,\Psi_n\right)\right\|_{A,I_p}}\right)$$

$$G_3(x,\varepsilon,t)=\sum_{n=1}^{\infty}\left(\frac{\Psi_n(x)X_n(\varepsilon)\sin\Omega_n t}{\Omega_n\left\|\left(X_n,\Psi_n\right)\right\|_{A,I_p}}\right),G_4(x,\varepsilon,t)=\sum_{n=1}^{\infty}\left(\frac{\Psi_n(x)\Psi_n(\varepsilon)\sin\Omega_n t}{\Omega_n\left\|\left(X_n,\Psi_n\right)\right\|_{A,I_p}}\right)$$

are the Green's functions.