Proceedings of the International Congress on Ultrasonics, Vienna, April 9-13, 2007, Paper ID 1162, Session R30: Ultrasound in anisotropic materials
doi:10.3728/ICUltrasonics.2007.Vienna.1162_shatalov

Non-axisymmetric vibrations of a transversely isotropic piezoelectric cylinder with different types of electric boundary conditions

Michael Y. Shatalov*, Arthur G. Every**, Alfred S. Yenwong –Fai***

*Sensor Science and Technology (SST) of CSIR Material Science and Manufacturing,
P.O. Box 395, Pretoria 0001, CSIR, South Africa and Department of Mathematics and Statistics
P.B.X680, Pretoria 0001 FIN-40014 Tshwane University of Technology, South Africa
e-mail: mshatlov@csir.co.za
**School of Physics, University of Witwatersrand, Po Box 2050, Johannesburg, South Africa,
e-mail: arthur.every@wits.ac.za
***Sensor Science and Technology (SST) of CSIR Material Science and Manufacturing,
P.O. Box 395, Pretoria 0001, CSIR, South Africa
e-mail: ayenwongfai@csir.co.za

Abstract: Coupled electro-mechanical non-axisymmetric vibrations are considered in a transversely isotropic piezoelectric cylinder with axial polarization. Solutions of the system of equations are found by exact integration using a seven functional method. The dispersion curves are plotted for propagating waves for non-axisymmetric vibrations of the cylinder. It is shown that the dispersion curves are sensitive to the form of electric boundary conditions.

Key words: piezoelectric cylinder, exact integration, electric boundary conditions.

A. Geometry of system

The coordinates r, θ and z are used to define the displacement vector components u, v and w which are depicted in the figure below.

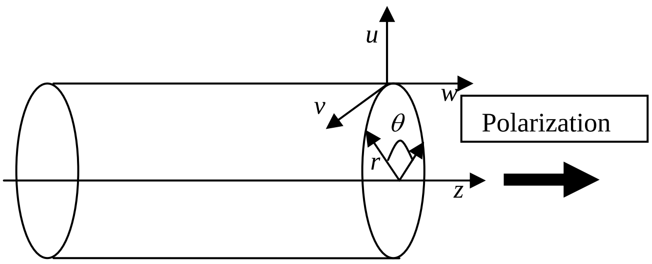

Fig. 1. Geometry and polarization direction of the piezoelectric cylinder

B. Basic equations

In this section we present the field equations satisfied in the material based on the following assumptions and approximations: linear elasticity, linear electromechanical coupling, quasi static approximation of the electric field, no free charges in the material and neglecting thermal effects and body forces. The relevant piezoelectric field equations are:

(i) The momentum equation and Gauss' law

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{\partial\sigma_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=\rho\ddot{u},$$

$$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\partial\sigma_{z\theta}}{\partial z}+\frac{2\sigma_{r\theta}}{r}=\rho\ddot{v},$$

$$\frac{\partial\sigma_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{\sigma_{rz}}{r}=\rho\ddot{w},$$

$$\frac{\partial D_r}{\partial r}+\frac{1}{r}\left(D_r+\frac{\partial D_\theta}{\partial\theta}\right)+\frac{\partial D_z}{\partial z}=0 \tag{1}$$

(ii) The coupled constitutive equations

$$\sigma_{rr}=c_{11}^E S_{rr}+c_{12}^E S_{\theta\theta}+c_{13}^E S_{zz}-e_{31}E_z;$$

$$\sigma_{\theta\theta}=c_{12}^E S_{rr}+c_{11}^E S_{\theta\theta}+c_{13}^E S_{zz}-e_{31}E_z;$$

$$\sigma_{zz}=c_{13}^E\left(S_{rr}+S_{\theta\theta}\right)+c_{33}^E S_{zz}-e_{33}E_z;$$

$$\sigma_{\theta z}=c_{55}^E S_{\theta z}-e_{15}E_\theta;\quad \sigma_{rz}=c_{55}^E S_{rz}-e_{15}E_r;$$

$$\sigma_{r\theta}=c_{66}^E S_{r\theta};$$

$$D_r=\varepsilon_1^S E_r+e_{15}S_{rz};$$

$$D_\theta=\varepsilon_1^S E_\theta+e_{15}S_{\theta z};$$

$$D_z=\varepsilon_3^S E_z+e_{31}\left(S_{rr}+S_{\theta\theta}\right)+e_{33}S_{zz} \tag{2}$$

(iii) The strain-mechanical displacement relations and electric field-electric potential relations

$$S_{rr}=\frac{\partial u}{\partial r};\quad S_{\theta\theta}=\frac{1}{r}\left(u+\frac{\partial v}{\partial\theta}\right);\quad S_{zz}=\frac{\partial w}{\partial z};\quad S_{rz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r};$$

$$S_{r\theta}=\frac{\partial v}{\partial r}+\frac{1}{r}\left(\frac{\partial u}{\partial\theta}-v\right);\quad S_{\theta z}=\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial\theta} \tag{3}$$

$$E_r=-\frac{\partial\phi}{\partial r};\quad E_\theta=-\frac{1}{r}\frac{\partial\phi}{\partial\theta};\quad E_z=-\frac{\partial\phi}{\partial z} \tag{4}$$

doi:10.3728/ICUltrasonics.2007.Vienna.1162_shatalov

C. Solutions of the equations

We seek harmonic wave solutions of the form

$$u=\left[\frac{\partial\Phi}{\partial r}+\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right]e^{i(kz-\omega t)};\quad v=\left[\frac{1}{r}\frac{\partial\Phi}{\partial\theta}-\frac{\partial\Psi}{\partial r}\right]e^{i(kz-\omega t)};$$

$$w=\eta\,\Phi\, e^{i(kz-\omega t)};\quad \phi=T(r,\theta)e^{i(kz-\omega t)} \tag{5}$$

where Φ, and Ψ are mechanical displacement potentials; η is an arbitrary constant to be determined in the analysis. Substituting (5) into (1) leads to the following equations in terms of Φ, T and Ψ

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left\{c_{11}^E\Delta\Phi+\left[\rho\omega^2-k^2c_{44}^E+i\eta k\left(c_{13}^E+c_{44}^E\right)\right]\Phi+ik\left(e_{15}+e_{31}\right)T\right\}-\frac{\partial}{\partial z}\left\{c_{66}^E\Delta\Psi-\left(\rho\omega^2-k^2c_{44}^E\right)\Psi\right\}=0$$

$$\left\{\left[ik\left(c_{13}^E+c_{44}^E\right)+\eta c_{44}^E\right]\Delta\Phi+\left(\rho\omega^2-k^2c_{33}^E\right)\eta\,\Phi\right\}+\left\{e_{15}\Delta T-e_{33}k^2T\right\}=0$$

$$\left\{\left[ik\left(e_{31}+e_{15}\right)+\eta e_{15}\right]\Delta\Phi-k^2e_{33}\eta\,\Phi\right\}-\left\{\varepsilon_1^S\Delta T-\varepsilon_3k^2T\right\}=0 \tag{6}$$

where $\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ - Laplace operator in the polar coordinates.

Let us assume that Φ and T satisfy the Helmholtz equations $\Delta\Phi+\xi^2\Phi=0$, $\Delta T+\xi^2T=0$, where ξ^2 is an unknown parameter which will be found later. We then observe that the following equations are satisfied:

$$T=\mu\Phi;\qquad \mu=\frac{\eta k^2e_{33}+\xi\left[(\eta+ik)e_{15}+ike_{31}\right]}{c_{11}\xi^2+c_{33}k^2},$$

$$\eta=\frac{-i}{k}\frac{\xi^4c_{11}^Ee_{15}+\xi^2\left[k^2\left(c_{11}^Ee_{33}-c_{13}^Ee_{15}-\left(c_{13}^E+c_{44}^E\right)e_{31}\right)-\rho\omega^2e_{15}\right]+k^2e_{33}\left(k^2c_{44}^E-\rho\omega^2\right)}{k^2\left[\left(c_{13}^E+c_{44}^E\right)e_{33}-c_{33}^E\left(e_{15}+e_{31}\right)\right]-\xi^2\left(c_{44}^Ee_{31}-c_{13}^Ee_{15}\right)+\rho\omega^2\left(e_{15}+e_{31}\right)}, \tag{7}$$

where, ξ satisfies the bi-cubic equation $\left(\xi^2\right)^3+a\left(\xi^2\right)^2+b\left(\xi^2\right)+c=0$, where

$$a=\frac{-\rho\omega^2\left[e_{15}^2+\left(c_{11}^E+c_{44}^E\right)\varepsilon_1^S\right]+k^2\varepsilon_1^S\left(c_{11}^Ec_{33}^E-2c_{13}^Ec_{44}^E-c_{13}^{E\,2}\right)+k^2\left[c_{44}^E\left(e_{31}^2+c_{11}^E\varepsilon_3^S\right)-2\left(c_{11}^Ee_{33}-c_{13}^Ee_{31}\right)+c_{13}^Ee_{15}^2\right]}{c_{11}^E\left(e_{15}^2+c_{44}^E\varepsilon_1^S\right)},$$

$$b=\frac{\begin{gathered}\varepsilon_1^S\rho^2\omega^4-k^2\rho\omega^2\left[\left(c_{33}^E+c_{44}^E\right)\varepsilon_1^S+\left(c_{11}^E+c_{44}^E\right)+\left(e_{15}+e_{31}\right)^2+2e_{15}e_{31}\right]+\\ k^4\left\{c_{33}^E\left[c_{11}^E\varepsilon_3^S+c_{44}^E\varepsilon_1^S+\left(e_{15}+e_{31}\right)^2\right]-2c_{13}^E\left[c_{44}^E\varepsilon_3^S+e_{33}\left(e_{15}+e_{31}\right)\right]+c_{11}^Ee_{33}^2-2c_{44}^Ee_{31}e_{33}-c_{13}^{E\,2}\varepsilon_3^S\right\}\end{gathered}}{c_{11}^E\left(e_{15}^2+c_{44}^E\varepsilon_1^S\right)},$$

$$c=\frac{k^2\left(k^2c_{44}^E-\rho\omega^2\right)\left[k^2\left(c_{44}^E\varepsilon_3^S+e_{33}^2\right)-\varepsilon_3^S\rho\omega^2\right]}{c_{11}^E\left(c_{44}^E\varepsilon_1^S+e_{15}^2\right)} \tag{8}$$

After solving the bi-cubic equation for ξ, we obtain the solution of the Helmholtz's equations:

$$\Phi_i(r,\theta)=A_iJ_m\left(\xi_ir\right)\cos(m\theta)\ (i=1,2,3),\qquad \Psi(r,\theta)=A_4J_m\left(\sqrt{\frac{\rho\omega^2-k^2c_{44}^E}{c_{66}^E}}\cdot r\right)\sin(m\theta) \tag{9}$$

And finally the displacements and electric potential are:

$$u=\left(\frac{\partial\Phi_1}{\partial r}+\frac{\partial\Phi_2}{\partial r}+\frac{\partial\Phi_3}{\partial r}+\frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right)e^{i(kz-\omega t)},\quad v=\left[\frac{1}{r}\left(\frac{\partial\Phi_1}{\partial\theta}+\frac{\partial\Phi_2}{\partial\theta}+\frac{\partial\Phi_3}{\partial\theta}\right)-\frac{\partial\Psi}{\partial r}\right]e^{i(kz-\omega t)},$$

$$w=\left(\eta_1\Phi_1+\eta_2\Phi_2+\eta_3\Phi_3\right)e^{i(kz-\omega t)},\qquad \phi=\left(\mu_1\Phi_1+\mu_2\Phi_2+\mu_3\Phi_3\right)e^{i(kz-\omega t)} \tag{10}$$

doi:10.3728/ICUltrasonics.2007.Vienna.1162_shatalov

D. Characteristic equation

The characteristic equation is obtained by imposing the following physically allowed boundary conditions at the surface of the cylinder :

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0 \text{ and } D = 0 \text{ or } \phi = 0 \text{ at } r = a \qquad (11)$$

This completes the analytical formulation of the problem.

E. Numerical results and discussions

In this section sample results for a PZT-4 composite are illustrated in the figures below for four cases; where the vertical and horizontal axes stand for the angular frequency ω and wave number-thickness product ka respectively and both in SI units. The ranges of ka and ω are: 0 to 6 and 0 to $2\pi \cdot 3000$ s^{-1} respectively.

Figure 2 represents the classical results, for waves with $m = 1$ in a transversely isotropic rod without electro-mechanical coupling, except for the straight line which goes through the origin, which is an artefact. In simulations we used a pure transversely isotropic material with the same elastic stiffness constants as for PZT-4.

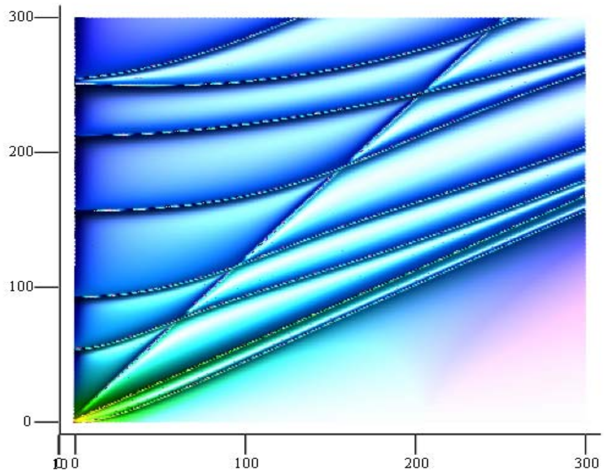

Fig. 2. Transversely isotropic cylinder ($m = 1$)

In Figures 3 and 4 the coupling constants have been reduced by a factor of five, the wave modes approach the classical results of Figure 2; however, there are three extra lines which are also artefacts. These artefacts depend on the method of solution and do not reveal the nature of dispersion curves.

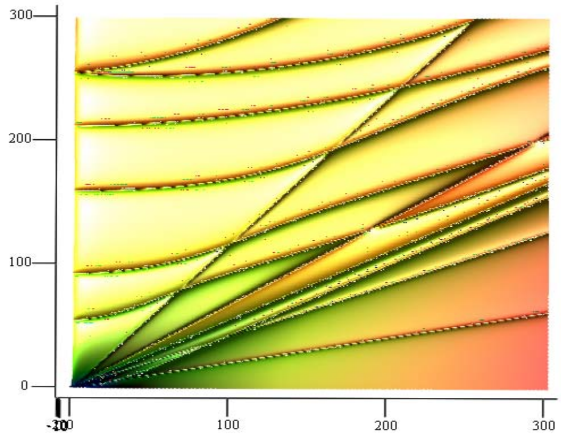

Fig. 3. ($D_3 = 0$, $m = 1$)

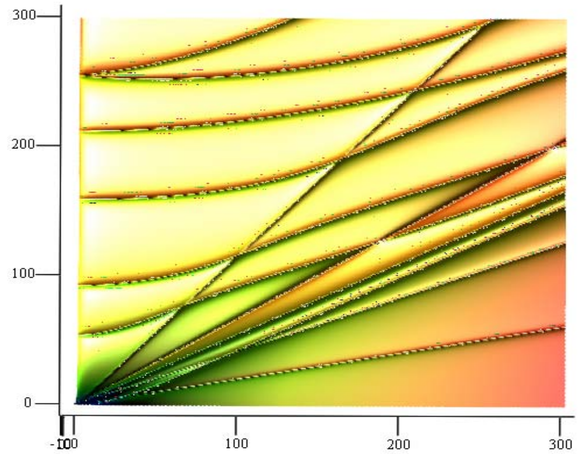

Fig. 4. ($\phi = 0$, $m = 1$)

Finally in Figures 5 and 6, where the standard coupling coefficients of PZT-4 are used, the wave modes are sensitive to the electrical boundary conditions. This sensitivity is especially pronounced for the higher modes and reveals the intersection of some dispersion curves for the case $[D_r]_{r=a} = 0$ (Fig. 5). This intersection is absent for $[\phi]_{r=a} = 0$ (Fig. 6).

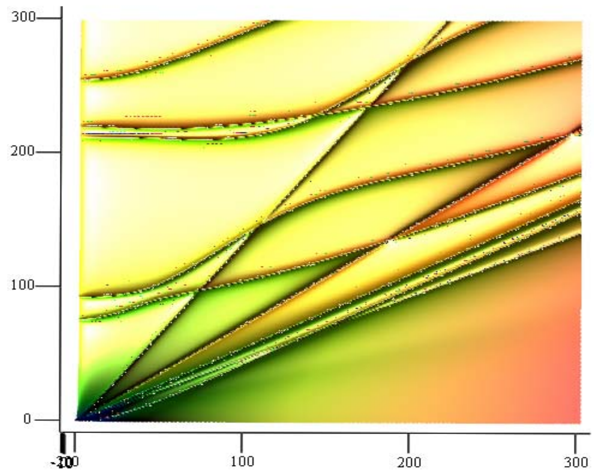

Fig. 5. ($D_3 = 0$, $m = 1$)

doi:10.3728/ICUltrasonics.2007.Vienna.1162_shatalov

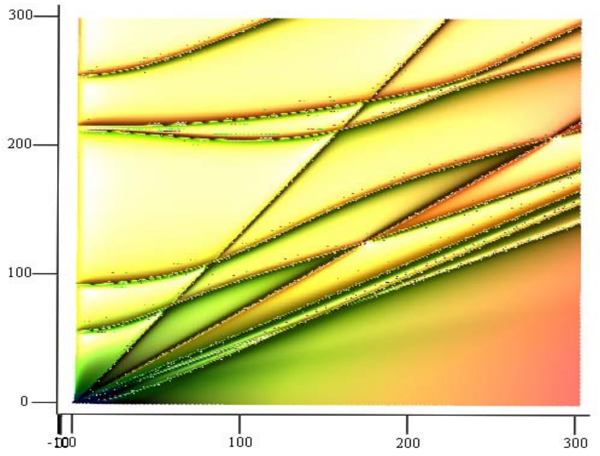

Fig. 6. ($\phi = 0$, $m = 1$)

F. Conclusions

The nature of electric boundary conditions plays an important role, in the study of propagating wave modes in a piezoelectric rod. This influence is revealed in the higher non-axisymmetric modes.

G. References

[1] C.Chree, Q.J. Math., 24: 340-354, 1890.
[2] I.Mirsky. J.Acoust.Soc.Am., 36: 2106-2122, 1964.
[3] M.J.Berliner, R.Solecki. J.Acoust.Soc.Am., 99(4): 1841-1847, 1996.
[4] A.H.Nayfeh, W.G.Abdelrahman, P.B.Nagy, J.Acoust.Soc.Am., 108(4): 1498-1504, 2000.