PRECESSION OF ELASTIC WAVES IN VIBRATING ISOTROPIC SPHERES AND TRANSVERSELY ISOTROPIC CYLINDERS SUBJECTED TO INERTIAL ROTATION

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Abstract

It was found by G. Bryan in 1890 that vibrating pattern of a rotating ring follows to a direction of the inertial rotation of this ring with an angular rate of the vibrating pattern smaller than the inertial rate. In 1979 E. Loper and D. Lynch proposed a hemispherical vibrating bell gyroscope utilising the Bryan's effect, which can measure an inertial angular rate and angle of rotation about the symmetry axis of the hemispherical shell. All these works exploited the precession properties of thin vibrating shells subjected to an inertial rotation around their axes of symmetry. In 1985 V. Zhuravlev generalized the abovementioned results and shown that the Bryan's effect has a three dimensional nature, i.e. that a vibrating pattern of an isotropic spherically symmetric body, arbitrary rotating in 3-D space, follows the inertial rotation of the solid body with a proportionality factor depending on the vibrating mode. This result had a qualitative nature without classification of vibrating modes and calculation of the corresponding proportionality factors. In the present paper radial and torsional vibrational modes are considered on the basis of an exact solution of 3-D equations of motion of an isotropic body in spherical coordinates. The solutions are obtained by means of a three potentials method in the spherical Bessel and associated Legendre functions. The proportionality factors of corresponding vibrating modes are calculated. The effects of gyroscopic forces on wave propagation in a transversely isotropic cylinder due to the inertial rotation are considered. The solutions are expressed in Bessel functions for different modes and corresponding Bryan's proportionality factors are calculated.

CONTENTS

- TRANSVERSELY ISOTROPIC CYLINDER.
- ISOTROPIC SPHERE.
- GYROSCOPIC EFFECTS IN VIBRATING AND ROTATING STRUCTURES.
- EXAMPLES.
- CONCLUSIONS.

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- **TRANSVERSELY ISOTROPIC CYLINDER.**
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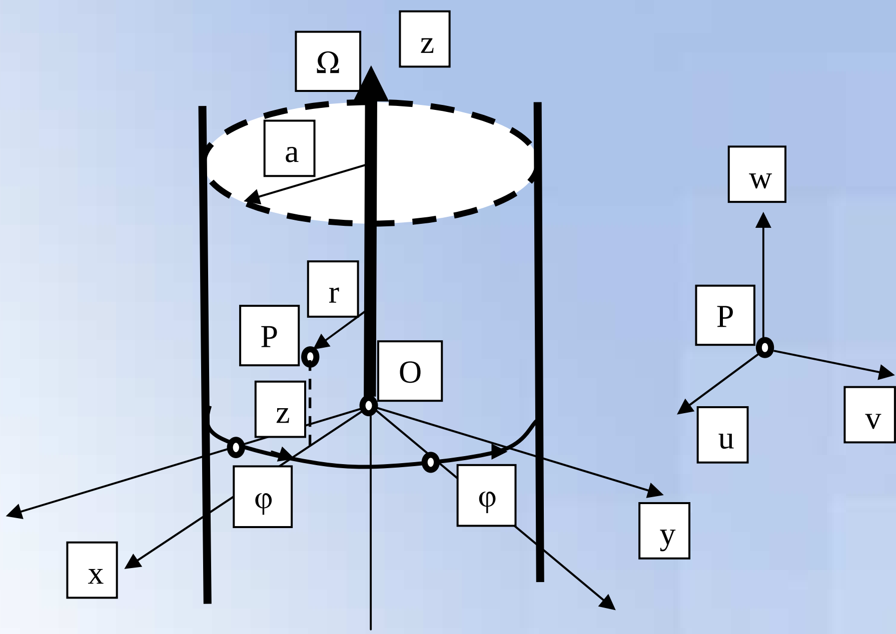
- **CONCLUSIONS.**

TRANSVERSELY ISOTROPIC CYLINDER - 1



$$\begin{cases} \dfrac{\partial \sigma_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{r\varphi}}{\partial \varphi} + \dfrac{\partial \sigma_{rz}}{\partial z} + \dfrac{1}{r}\left(\sigma_{rr} - \sigma_{\varphi\varphi}\right) = \rho \ddot{u} \\ \dfrac{\partial \sigma_{r\varphi}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \dfrac{\partial \sigma_{\varphi z}}{\partial z} + \dfrac{2}{r}\sigma_{r\varphi} = \rho \ddot{v} \\ \dfrac{\partial \sigma_{rz}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{\varphi z}}{\partial \varphi} + \dfrac{\partial \sigma_{zz}}{\partial z} + \dfrac{1}{r}\sigma_{rz} = \rho \ddot{w} \end{cases}$$

TRANSVERSELY ISOTROPIC CYLINDER - 2

$$\begin{cases} \dfrac{\partial \sigma_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{r\varphi}}{\partial \varphi} + \dfrac{\partial \sigma_{rz}}{\partial z} + \dfrac{1}{r}\left(\sigma_{rr} - \sigma_{\varphi\varphi}\right) = \rho \ddot{u} \\ \dfrac{\partial \sigma_{r\varphi}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \dfrac{\partial \sigma_{\varphi z}}{\partial z} + \dfrac{2}{r}\sigma_{r\varphi} = \rho \ddot{v} \\ \dfrac{\partial \sigma_{rz}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \sigma_{\varphi z}}{\partial \varphi} + \dfrac{\partial \sigma_{zz}}{\partial z} + \dfrac{1}{r}\sigma_{rz} = \rho \ddot{w} \end{cases}$$

$$\sigma_{rr} = c_{11}\varepsilon_{rr} + c_{12}\varepsilon_{\varphi\varphi} + c_{13}\varepsilon_{zz}; \quad \sigma_{\varphi\varphi} = c_{12}\varepsilon_{rr} + c_{11}\varepsilon_{\varphi\varphi} + c_{13}\varepsilon_{zz};$$

$$\sigma_{zz} = c_{13}\varepsilon_{rr} + c_{13}\varepsilon_{\varphi\varphi} + c_{33}\varepsilon_{zz}; \quad \sigma_{\varphi z} = c_{44}\varepsilon_{\varphi z}; \sigma_{rz} = c_{44}\varepsilon_{rz}; \sigma_{\varphi r} = c_{66}\varepsilon_{\varphi r}$$

$$\left(c_{66} = \frac{c_{11} - c_{12}}{2}\right)$$

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}; \quad \varepsilon_{\varphi\varphi} = \frac{1}{r}\left(u + \frac{\partial v}{\partial \varphi}\right); \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}; \quad \varepsilon_{rz} = \varepsilon_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

$$\varepsilon_{r\varphi} = \varepsilon_{\varphi r} = \frac{\partial v}{\partial r} + \frac{1}{r}\left(\frac{\partial u}{\partial \varphi} - v\right); \quad \varepsilon_{\varphi z} = \varepsilon_{z\varphi} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \varphi}$$

TRANSVERSELY ISOTROPIC CYLINDER - 3

$$u=\left[\frac{\partial\,\mathrm{X}(r,\varphi)}{\partial r}+\frac{1}{r}\frac{\partial\,\Psi(r,\varphi)}{\partial\varphi}\right]\cdot e^{j\left(\omega t-\hat{k}z\right)};\quad v=\left[\frac{1}{r}\frac{\partial\,\mathrm{X}(r,\varphi)}{\partial\varphi}-\frac{\partial\,\Psi(r,\varphi)}{\partial r}\right]\cdot e^{j\left(\omega t-\hat{k}z\right)};\quad w=\eta\cdot\mathrm{X}(r,\varphi)\cdot e^{j\left(\omega t-\hat{k}z\right)}$$

$$\nabla^2\mathrm{X}_1+\varsigma_1{}^2\mathrm{X}_1=0;\qquad \nabla^2\mathrm{X}_2+\varsigma_2{}^2\mathrm{X}_2=0;\qquad \nabla^2\Psi+\varsigma_3{}^2\Psi=0$$

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\varphi^2}$$

$$\varsigma_{1,2}=\sqrt{\frac{B\pm\sqrt{B^2-4AC}}{2A}};\quad \zeta_3=\sqrt{\frac{\rho\omega^2-\hat{k}^2c_{44}}{c_{66}}}$$

$$A=c_{11}c_{44};\quad B=\left(c_{11}+c_{44}\right)\rho\omega^2+\left(c_{13}{}^2+2c_{13}c_{44}-c_{11}c_{33}\right)\hat{k}^2;\quad C=\left(\rho\omega^2-\hat{k}^2c_{33}\right)\left(\rho\omega^2-\hat{k}^2c_{44}\right)$$

$$\mathrm{X}_1(r,\varphi)=R_1(r)\cdot\Phi_1(\varphi);\quad \mathrm{X}_2(r,\varphi)=R_2(r)\cdot\Phi_2(\varphi);\quad \Psi(r,\varphi)=R_3(r)\cdot\Phi_3(\varphi)$$

TRANSVERSELY ISOTROPIC CYLINDER - 4

$$\mathrm{X}_1(r,\varphi)=R_1(r)\cdot\Phi_1(\varphi);\quad \mathrm{X}_2(r,\varphi)=R_2(r)\cdot\Phi_2(\varphi);\quad \Psi(r,\varphi)=R_3(r)\cdot\Phi_3(\varphi)$$

$$\mathrm{X}^{(m)}{}_1(r,\varphi)=A^{(m)}{}_1J_m(\varsigma_1 r)\cos(m\varphi);\quad \mathrm{X}^{(m)}{}_2(r,\varphi)=A^{(m)}{}_2J_m(\varsigma_2 r)\cos(m\varphi);\quad \Psi^{(m)}(r,\varphi)=A^{(m)}{}_3J_m(\varsigma_3 r)\sin(m\varphi)$$

$$u^{(m)}(r,\varphi,z,t)=\left[\frac{\partial\,\mathrm{X}^{(m)}{}_1(r,\varphi)}{\partial r}+\frac{\partial\,\mathrm{X}^{(m)}{}_2(r,\varphi)}{\partial r}+\frac{1}{r}\frac{\partial\,\Psi(r,\varphi)}{\partial\varphi}\right]\cdot e^{i\left(\omega t-\hat{k}z\right)};$$

$$v^{(m)}(r,\varphi,z,t)=\left[\frac{1}{r}\frac{\partial\,\mathrm{X}^{(m)}{}_1(r,\varphi)}{\partial\varphi}+\frac{1}{r}\frac{\partial\,\mathrm{X}^{(m)}{}_2(r,\varphi)}{\partial\varphi}-\frac{\partial\,\Psi(r,\varphi)}{\partial r}\right]\cdot e^{i\left(\omega t-\hat{k}z\right)};$$

$$w^{(m)}(r,\varphi,z,t)=\left[\eta_1\cdot\mathrm{X}^{(m)}{}_1(r,\varphi)+\eta_2\cdot\mathrm{X}^{(m)}{}_2(r,\varphi)\right]\cdot e^{i\left(\omega t-\hat{k}z+\frac{\pi}{2}\right)}$$

$$\eta_1=\frac{c_{11}\xi_1{}^2+\left(\hat{k}^2c_{44}-\rho\omega^2\right)}{\hat{k}\left(c_{13}+c_{44}\right)};\qquad \eta_2=\frac{c_{11}\varsigma_2{}^2+\left(\hat{k}^2c_{44}-\rho\omega^2\right)}{\hat{k}\left(c_{13}+c_{44}\right)}$$

$$u(r,\varphi,z,t)=\sum_{m=0}^{\infty}u^{(m)}(r,\varphi,z,t);\quad v(r,\varphi,z,t)=\sum_{m=1}^{\infty}v^{(m)}(r,\varphi,z,t);\quad w(r,\varphi,z,t)=\sum_{m=0}^{\infty}w^{(m)}(r,\varphi,z,t)$$

Boundary conditions: $\left[\sigma_{rr}\right]_{r=a}=\left[\sigma_{rz}\right]_{r=a}=\left[\sigma_{r\varphi}\right]_{r=a}=0$

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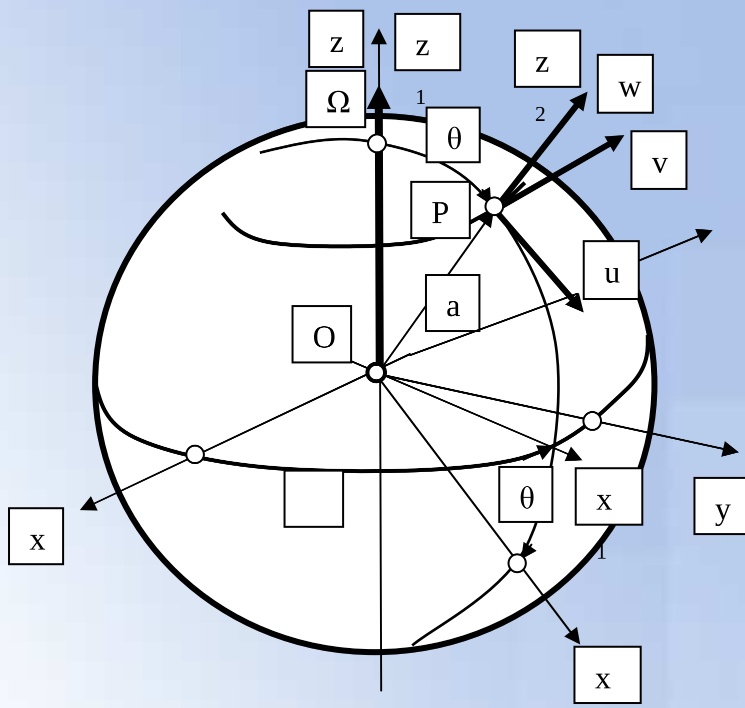
ISOTROPIC SPHERE - 1



$$
\begin{cases}
\dfrac{\partial\ \sigma_{rr}}{\partial r}+\dfrac{1}{r}\dfrac{\partial\ \sigma_{r\theta}}{\partial \theta}+\dfrac{1}{r\sin\theta}\dfrac{\partial\ \sigma_{r\varphi}}{\partial \varphi}+\dfrac{2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{\varphi\varphi}+\cot\theta\cdot\sigma_{r\theta}}{r}=\rho\dfrac{\partial^2 w}{\partial t^2} \\
\dfrac{1}{r}\dfrac{\partial\ \sigma_{\theta\theta}}{\partial \theta}+\dfrac{\partial\ \sigma_{r\theta}}{\partial r}+\dfrac{1}{r\sin\theta}\dfrac{\partial\ \sigma_{\theta\varphi}}{\partial \varphi}+\dfrac{3\sigma_{r\theta}+\cot\theta\cdot\left(\sigma_{\theta\theta}-\sigma_{\varphi\varphi}\right)}{r}=\rho\dfrac{\partial^2 u}{\partial t^2} \\
\dfrac{1}{r\sin\theta}\dfrac{\partial\ \sigma_{\varphi\varphi}}{\partial \varphi}+\dfrac{\partial\ \sigma_{r\varphi}}{\partial r}+\dfrac{1}{r}\dfrac{\partial\ \sigma_{\theta\varphi}}{\partial \theta}+\dfrac{3\sigma_{r\varphi}+2\cot\theta\cdot\sigma_{\theta\varphi}}{r}=\rho\dfrac{\partial^2 v}{\partial t^2}
\end{cases}
$$

ISOTROPIC SPHERE - 2

$$\begin{cases} \dfrac{\partial\ \sigma_{rr}}{\partial r}+\dfrac{1}{r}\dfrac{\partial\ \sigma_{r\theta}}{\partial\theta}+\dfrac{1}{r\sin\theta}\dfrac{\partial\ \sigma_{r\varphi}}{\partial\varphi}+\dfrac{2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{\varphi\varphi}+\cot\theta\cdot\sigma_{r\theta}}{r}=\rho\dfrac{\partial^2 w}{\partial t^2} \\ \dfrac{1}{r}\dfrac{\partial\ \sigma_{\theta\theta}}{\partial\theta}+\dfrac{\partial\ \sigma_{r\theta}}{\partial r}+\dfrac{1}{r\sin\theta}\dfrac{\partial\ \sigma_{\theta\varphi}}{\partial\varphi}+\dfrac{3\sigma_{r\theta}+\cot\theta\cdot\left(\sigma_{\theta\theta}-\sigma_{\varphi\varphi}\right)}{r}=\rho\dfrac{\partial^2 u}{\partial t^2} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial\ \sigma_{\varphi\varphi}}{\partial\varphi}+\dfrac{\partial\ \sigma_{r\varphi}}{\partial r}+\dfrac{1}{r}\dfrac{\partial\ \sigma_{\theta\varphi}}{\partial\theta}+\dfrac{3\sigma_{r\varphi}+2\cot\theta\cdot\sigma_{\theta\varphi}}{r}=\rho\dfrac{\partial^2 v}{\partial t^2} \end{cases}$$

$$\sigma_{rr}=\lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\varphi\varphi}\right)+2\mu\varepsilon_{rr};\qquad \sigma_{\theta\theta}=\lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\varphi\varphi}\right)+2\mu\varepsilon_{\theta\theta};$$

$$\sigma_{\varphi\varphi}=\lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\varphi\varphi}\right)+2\mu\varepsilon_{\varphi\varphi};\quad \sigma_{r\theta}=\mu\varepsilon_{r\theta};\quad \sigma_{r\varphi}=\mu\varepsilon_{r\varphi};\quad \sigma_{\theta\varphi}=\mu\varepsilon_{\theta\varphi}$$

$$\varepsilon_{rr}=w'_r;\qquad \varepsilon_{\theta\theta}=\frac{1}{r}(u'_\theta+w);\qquad \varepsilon_{\varphi\varphi}=\frac{1}{r}\left(\cot\theta\cdot u+\frac{1}{\sin\theta}v'_\varphi+w\right);$$

$$\varepsilon_{r\theta}=u'_r+\frac{1}{r}(-u+w'_\theta);\quad \varepsilon_{r\varphi}=v'_r+\frac{1}{r}\left(-v+\frac{1}{\sin\theta}w'_\varphi\right);\quad \varepsilon_{\theta\varphi}=\frac{1}{r}\left(\frac{1}{\sin\theta}u'_\varphi-\cot\theta\cdot v+v'_\theta\right)$$

$$(u,v,w)\to(\Phi,\Psi,\mathrm{X})$$

$$w=\Phi'_r+r\cdot\left[\left(\mathrm{X}''_{rr}+\frac{2}{r}\mathrm{X}'_r\right)-\nabla^2\mathrm{X}\right];\quad u=\left[\mathrm{X}'_r+\frac{1}{r}(\Phi+\mathrm{X})\right]'_\theta+\frac{1}{a\sin\theta}\Psi'_\varphi;\quad v=\frac{1}{\sin\theta}\cdot\left[\mathrm{X}'_r+\frac{1}{r}(\Phi+\mathrm{X})\right]'_\varphi-\frac{1}{a}\Psi'_\theta$$

ISOTROPIC SPHERE - 3

$$(\lambda+2\mu)\cdot\nabla^2\Phi=\rho\ddot{\Phi};\qquad \mu\cdot\nabla^2\Psi=\rho\ddot{\Psi};\qquad \mu\cdot\nabla^2\mathrm{X}=\rho\ddot{\mathrm{X}}$$

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\left(\frac{d}{dt}\rightarrow i\,\omega,\quad \frac{d^2}{dt}\rightarrow-\omega^2\right)$$

$$\begin{cases}\Phi_{mn}(r,\theta,\varphi)=A_{mn}\cdot j_n(k_1r)\cdot P_n^m(\cos\theta)\cdot\cos(m\varphi)\\ \mathrm{X}_{mn}(r,\theta,\varphi)=B_{mn}\cdot j_n(k_2r)\cdot P_n^m(\cos\theta)\cdot\cos(m\varphi)\\ \Psi_{mn}(r,\theta,\varphi)=D_{mn}\cdot j_n(k_2r)\cdot P_n^m(\cos\theta)\cdot\sin(m\varphi)\end{cases}$$

$$k_1=k_1(\omega)=\omega/c_1\quad c_1=\sqrt{(\lambda+2\mu)/\rho}\qquad k_2=k_2(\omega)=\omega/c_2\quad c_2=\sqrt{\mu/\rho}$$

Boundary conditions: $\left[\sigma_{rr}\right]_{r=a}=\left[\sigma_{r\theta}\right]_{r=a}=\left[\sigma_{r\varphi}\right]_{r=a}=0$

ISOTROPIC SPHERE - 4

Spheroidal modes:

$$\begin{cases} \left\{ \left[\Phi''_{rr} - \dfrac{k_2^2(\omega)}{2} \cdot \dfrac{\lambda}{\lambda + 2\mu} \cdot \Phi \right] + \dfrac{2}{r} \cdot \left[k_2^2(\omega) \cdot (r\mathrm{X}'_r + \mathrm{X}) + (r\mathrm{X}'_r + \mathrm{X})''_{rr} \right] \right\}_{r=a} = 0 \\ \left\{ \dfrac{1}{r} \cdot \left[\Phi'_r - \dfrac{1}{r} \cdot \Phi \right] + \left[\mathrm{X}''_{rr} + \dfrac{1}{r} \cdot \mathrm{X}'_r + \left(\dfrac{k_2^2(\omega)}{2} - \dfrac{1}{r^2} \right) \cdot \mathrm{X} \right] \right\}_{r=a} = 0 \end{cases}$$

Torsional modes:

$$\left[\Psi'_r - \frac{1}{r} \cdot \Psi \right]_{r=a} = 0$$

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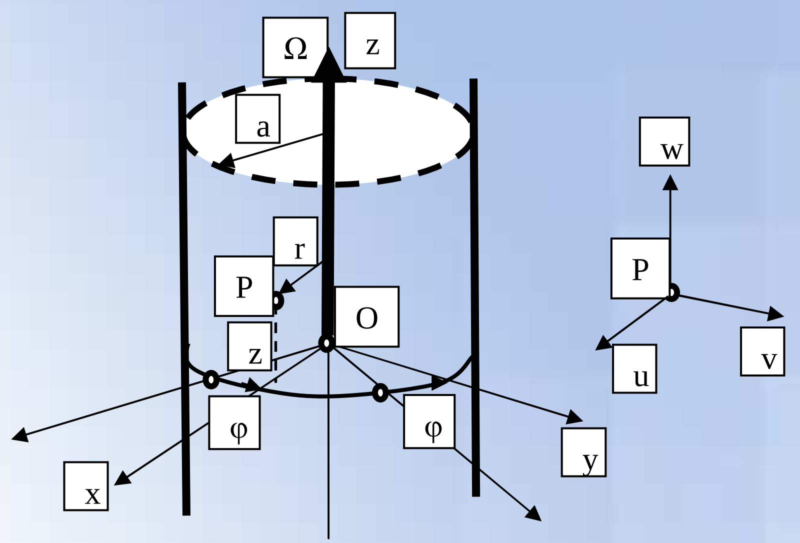
GYROSCOPIC EFFECTS IN VIBRATING AND ROTATING STRUCTURES - 1

Lagrangian of a Solid Cylinder - 1



$$\vec{V} = \dot{\vec{r}} + \vec{\Omega} \times \vec{r} = \begin{bmatrix} \dot{u} - \Omega v \\ \dot{v} + \Omega(r+u) \\ \dot{w} \end{bmatrix}$$

$$u_m = U_m(r) \cdot \left[C_m(t) \cos m\varphi + S_m(t) \sin m\varphi \right]$$
$$v_m = V_m(r) \cdot \left[S_m(t) \cos m\varphi - C_m(t) \sin m\varphi \right]$$
$$w_m = W_m(r) \cdot \left[C_m(t) \cos m\varphi + S_m(t) \sin m\varphi \right]$$

GYROSCOPIC EFFECTS IN VIBRATING AND ROTATING STRUCTURES - 2

Lagrangian of a Solid Cylinder - 2

Kinetic energy:

$$T = \frac{\rho}{2}\int_0^{2\pi}\int_0^{a}\left\|\vec{V}\right\|^2 dr\, d\varphi = T\left(\dot{C},\dot{S},C,S\right) \approx \frac{1}{2} I_0 \cdot \left(\dot{C}^2 + \dot{S}^2\right) + \Omega \cdot I_1 \cdot \left(C\dot{S} - \dot{C}S\right)$$

$$I_0 = \frac{\rho}{2}\cdot\int_0^{a}\left[U^2(r) + V^2(r) + W^2(r)\right]\cdot r\, dr\,; \qquad I_1 = \rho \cdot \int_0^{\pi}\int_0^{a}\left[U(r)\cdot V(r)\right]\cdot r\, dr$$

Strain energy:

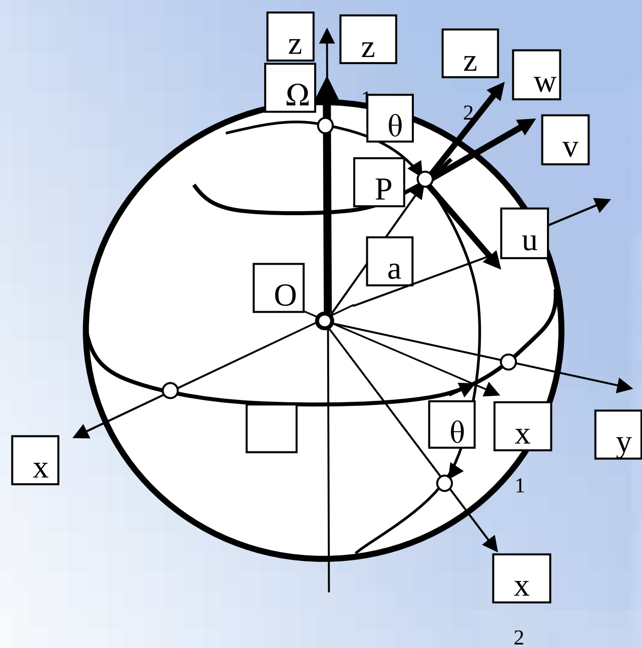
$$P(C,S) = \frac{1}{2}\int_0^{2\pi}\int_0^{a}\left(\sigma_{rr}\varepsilon_{rr} + \sigma_{\varphi\varphi}\varepsilon_{\varphi\varphi} + \sigma_{zz}\varepsilon_{zz} + \sigma_{r\varphi}\varepsilon_{r\varphi} + \sigma_{rz}\varepsilon_{rz} + \sigma_{z\varphi}\varepsilon_{z\varphi}\right)\cdot r\, dr\, d\varphi = \frac{1}{2} I_2 \left(C^2 + S^2\right)$$

Lagrangian:

$$L = L\left(\dot{C},\dot{S},C,S\right) = T\left(\dot{C},\dot{S},C,S\right) - P(C,S) \approx \frac{1}{2} I_0 \cdot \left(\dot{C}^2 + \dot{S}^2\right) + \Omega \cdot I_1 \cdot \left(C\dot{S} - \dot{C}S\right) - \frac{1}{2} I_2 \left(C^2 + S^2\right)$$

GYROSCOPIC EFFECTS IN VIBRATING AND ROTATING STRUCTURES - 3

Lagrangian of a Solid Sphere - 1



$$\vec{V} = \vec{\dot{r}} + \vec{\Omega} \times \vec{r} = \begin{bmatrix} \dot{u} - \Omega v \cos\theta \\ \dot{v} + \Omega\left[u \cos\theta + (r + w)\sin\theta\right] \\ \dot{w} - \Omega v \sin\theta \end{bmatrix}$$

$$u_{mn} = U_{mn}(r,\theta) \cdot \left[C_{mn}(t)\cos m\varphi + S_{mn}(t)\sin m\varphi\right]$$
$$v_{mn} = V_{mn}(r,\theta) \cdot \left[S_{mn}(t)\cos m\varphi - C_{mn}(t)\sin m\varphi\right]$$
$$w_{mn} = W_{mn}(r,\theta) \cdot \left[C_{mn}(t)\cos m\varphi + S_{mn}(t)\sin m\varphi\right]$$

GYROSCOPIC EFFECTS IN VIBRATING AND ROTATING STRUCTURES - 4

Lagrangian of a Solid Sphere - 2

Kinetic energy:

$$T = \frac{\rho}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left\|\vec{V}\right\|^2 r^2 \sin\theta \, dr \, d\theta \, d\varphi = T\left(\dot{C},\dot{S},C,S\right) \approx \frac{1}{2} I_0 \cdot \left(\dot{C}^2 + \dot{S}^2\right) + \Omega \cdot I_1 \cdot \left(C\dot{S} - \dot{C}S\right)$$

$$I_0 = \rho \cdot \int_0^{\pi}\int_0^{a}\left[U^2(r,\theta) + V^2(r,\theta) + W^2(r,\theta)\right] \cdot r^2 \cdot \sin\theta \, dr \, d\theta;$$

$$I_1 = 2\rho \cdot \int_0^{\pi}\int_0^{a}\left[\left(U(r,\theta)\cdot\cos\theta + W(r,\theta)\cdot\sin\theta\right)\cdot V(r,\theta)\right] \cdot r^2 \cdot \sin\theta \, dr \, d\theta$$

Strain energy:

$$P(C,S) = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left(\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{\varphi\varphi}\varepsilon_{\varphi\varphi} + \sigma_{r\theta}\varepsilon_{r\theta} + \sigma_{r\varphi}\varepsilon_{r\varphi} + \sigma_{\theta\varphi}\varepsilon_{\theta\varphi}\right) \cdot r^2 \sin\theta \, dr \, d\theta \, d\varphi = \frac{1}{2} I_2 \left(C^2 + S^2\right)$$

Lagrangian:

$$L = L\left(\dot{C},\dot{S},C,S\right) = T\left(\dot{C},\dot{S},C,S\right) - P(C,S) \approx \frac{1}{2} I_0 \cdot \left(\dot{C}^2 + \dot{S}^2\right) + \Omega \cdot I_1 \cdot \left(C\dot{S} - \dot{C}S\right) - \frac{1}{2} I_2 \left(C^2 + S^2\right)$$

GYROSCOPIC EFFECTS IN VIBRATING AND ROTATING STRUCTURES – 5

Equations of Motion - 1

Equations of motion:

$$\begin{cases} \ddot{C} - 2\eta\,\Omega\,\dot{S} + \omega^2 C = 0 \\ \ddot{S} + 2\eta\,\Omega\dot{C} + \omega^2 S = 0 \end{cases}$$

Bryan's Factor:

$$\eta = \frac{I_1}{I_0}$$

<u>Cylinder:</u>

$$I_0 = \frac{\rho}{2}\cdot\int_0^a \left[U^2(r)+V^2(r)+W^2(r)\right]\cdot r\,dr\,;$$

$$I_1 = \rho\cdot\int_0^\pi\int_0^a \left[U(r)\cdot V(r)\right]\cdot r\,dr$$

<u>Sphere:</u>

$$I_0 = \frac{\rho}{2}\cdot\int_0^\pi\int_0^a \left[U^2(r,\theta)+V^2(r,\theta)+W^2(r,\theta)\right]\cdot r^2\cdot\sin\theta\,dr\,d\theta;$$

$$I_1 = \rho\cdot\int_0^\pi\int_0^a \left[\left(U(r,\theta)\cdot\cos\theta+W(r,\theta)\cdot\sin\theta\right)\cdot V(r,\theta)\right]\cdot r^2\cdot\sin\theta\,dr\,d\theta$$

GYROSCOPIC EFFECTS IN VIBRATING AND ROTATING STRUCTURES – 6

Equations of Motion - 2

$$Z = C + iS$$

$$\ddot{Z} + 2i\eta\Omega\dot{Z} + \omega^2 Z = 0$$

$Z \rightarrow Y:$ $\quad Z(t) = Y(t)\cdot e^{i\alpha t}$ $\quad \alpha = const$

$$\dot{Z} = \left(\dot{Y} + i\alpha Y\right)\cdot e^{i\alpha t}; \qquad \ddot{Z} = \left(\ddot{Y} + 2i\alpha\dot{Y} - \alpha^2 Y\right)\cdot e^{i\alpha t}$$

$$\ddot{Y} + 2i(\alpha + \eta\Omega)\dot{Y} + \left(\omega^2 - \alpha^2 - 2\alpha\eta\Omega\right)Y = 0$$

$$\alpha = -\eta\Omega$$

$$\ddot{Y} + \omega^2 Y \approx 0 \qquad Z(t) = Y(t)\cdot e^{-i\eta\Omega t}$$

$$\tilde{\Omega} = -\eta\Omega \qquad \tilde{\Omega} = (1-\eta)\Omega$$

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EXAMPLE - 1

Sphere:

$$a = 0.5\,m \quad E = 7\cdot 10^{10}\ N/m^2 \quad \nu = 0.33 \quad \rho = 2.7\cdot 10^3\ kg/m^3$$

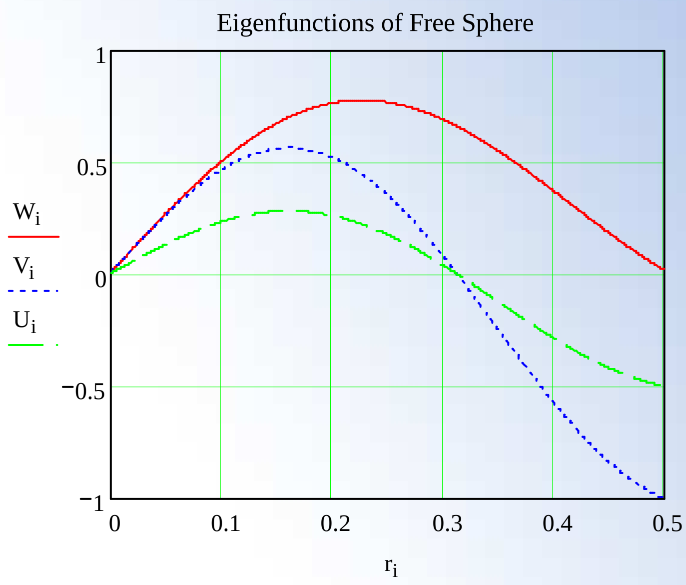
m=2	Eigenvalues (free boundary, Hz)	Bryan's factor (free boundary)	Eigenvalues (**Re**, acoustic medium, Hz)	Bryan's factor (acoustic medium)
n=2	2633	0.921	2567	0.969
	5056	0.137	5050	0.403
	8563	0.300	6616	0.333
			8190	0.350
	10848	0.270	9728	0.329
n=3	3924	0.515	2296	0.608
	6654	0.127	5660	0.175
			6654	0.174
	9910	0.136	7252	0.173
			8824	0.182

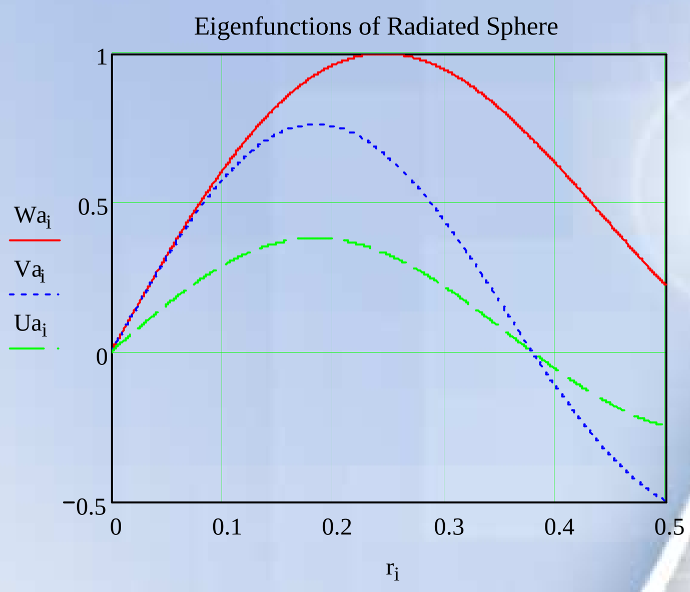
EXAMPLE - 2

m=3	Eigenvalues (free boundary, Hz)	Bryan's factor (free boundary)	Eigenvalues (**Re**, acoustic medium, Hz)	Bryan's factor (acoustic medium)
n=3	3924	0.634	2296	0.708
	6654	0.000	5660	0.149
			6654	0.151
	9910	0.073	7252	0.103
			8824	0.110

EXAMPLE - 3

Eigenfunctions, corresponding to m=2, n=2, f=5056 Hz:





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CONCLUSIONS

- **Expressions for Bryan's factor have been derived, which characterizes the coefficients of proportionality between angular rate of precession of a vibrating pattern and the inertial angular rate of an isotropic elastic body.**
- **It has been pointed out that the Bryan's factor is an invariant of sphere's radius, its mass density and modulus of elasticity; it depends on Poisson's ratio.**
- **In the rotating coordinate system the spheroidal vibrating patterns precess in the direction, which is opposite to the direction of inertial rotation (positive Bryan's factor); the torsional patterns precess in the direction of inertial rotation (positive Bryan's factor).**