APPROXIMATE ANALYSIS OF RIGID PLATE LOADING ON ELASTIC MULTI-LAYERED SYSTEMS

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ABSTRACT

GAMES software is well known for its capability to compute responses for uniformly distributed load acting on the surface of a multi-layered linear elastic system. In this study a method was developed to approximate rigid plate loading to be used in GAMES software. When a rigid plate load is acting on the surface of a multilayered system, the resulting interface contact stress distribution is not uniform. Since contact stress distribution between rigid plate and semi-infinite medium is well known, this distribution was approximated using uniformly distributed multiple loads and analysis performed using GAMES. Results have shown good agreement with the theory for the case of a semi-infinite medium. Furthermore, extension of this method to multilayered system has shown good results if the ratio of the first layer thickness over the loading plate radius is greater than 2.

KEY WORDS

rigid plate loading, elastic multilayered system, GAMES, axi-symmetric analysis

INTRODUCTION

Rigid plate loading test is generally used to evaluate load bearing capacity of pavement layers like subgrade, subbase and base layers. When structures like pavements are to be analyzed, most of the time the structural system is considered to be multilayered linear elastic. Furthermore, uniformly distributed circular load is assumed to act on the surface of the pavement and it is possible to use software like ELSA, CHEVRON, BISAR[1] as well as GAMES[2]. However, when a test like rigid plate loading is performed, the interface contact stresses are not uniformly distributed and software such as the ones listed above may not be used for analysis. Authors of this paper were unable, through literature search, to find any document describing the application of multilayer linear elastic theory to the rigid plate loading. Because of the facts highlighted above, this research evaluates the method of approximating theoretical solutions of rigid plate loading by using GAMES software.

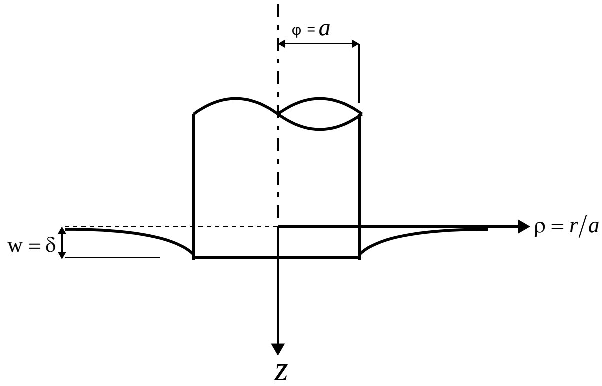


Figure 1 Deformation due to rigid body loading

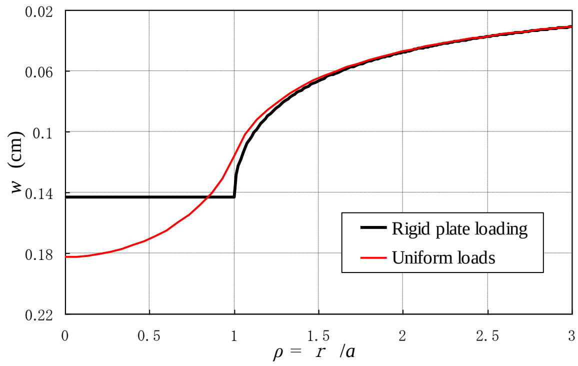


Figure 2 Surface deflections due to rigid plate and uniformly distributed loads ($P = 49\text{kN}, E_1 = 100\,\text{MPa}, \nu = 0.35$)

Sneddon derived the solutions for a semi-infinite medium subjected to circular rigid body loading as shown in **Figure 1**[3]. In this case, surface deflection under the base of the rigid body is constant and normal stresses at the surface but outside the base become zero. **Figure 2** was drawn to compare surface deflections from the applications of uniformly distributed loading and rigid plate loading. This figure clearly shows that in the vicinity of the load application area, deflections from the analysis of uniformly distributed loading are different from results of rigid body loading. This phenomenon is supported by Saint - Venant' principle, which states that there is high influence of load distribution near the point of application and this influence diminishes as you move away from the point of load application. It is clear from **Figure 2** that there is a very good match of deflection results for $r/a \geq 2$, there is a good match of deflection results.

In this research, distribution of contact stresses derived by Sneddon was approximated by multiple uniformly distributed loads, and responses for semi-infinite medium were analyzed using GAMES software. Accuracy of this approach was evaluated by comparing results obtained by Sneddon for a semi-infinite layer to results from using approximate stress distribution. Extension of this approach to a multilayered system was evaluated by checking whether deflections under the base of a rigid plate remained constant when layer moduli were varied for a two-layer system.

Contact stress distribution under rigid plate

Theoretical stress distribution

Assume a rigid plate load acting on the surface of a semi-infinite body to be considered an axi-symmetric problem, and then the boundary conditions can be represented as shown below:

$$w(r,0) = \delta \qquad 0 < r \le a \tag{1a}$$

$$\sigma_z(r,0) = 0 \qquad a < r \tag{1b}$$

δ is the amount of compression (surface deflection) and a is radius of the loading plate. According to Sneddon, shear modulus G, Poisson's ratio ν, and distribution of contact stresses on a semi-infinite medium $p(r)$ are related as shown in the following equation:

$$p(r) = -\frac{2G\delta}{(1-\nu)\pi}\frac{1}{\sqrt{a^2 - r^2}} \tag{2}$$

From equation (2), contact stress at the center of the load is $-2G\delta/((1-\nu)\pi a)$, while at the load edge, contact stresses becomes $-\infty$. Equation (2) can be integrated across the circular loading section to obtain the total load P as shown below:

$$P = \int_0^a \int_0^{2\pi} p(r) r d\theta dr = -\frac{4aG\delta}{1-\nu} \tag{3}$$

From equations (2) and (3), the relation between $p(r)$ and P can be obtained as follows:

$$p(r) = \frac{P}{2\pi a\sqrt{a^2 - r^2}} \tag{4}$$

Furthermore, Sneddon determined surface deflections using the following equation:

$$w(r,0) = \begin{cases} \delta & 0 \le r \le a \\ \dfrac{2\delta}{\pi}\sin^{-1}\dfrac{a}{r} & r > a \end{cases} \tag{5}$$

Approximate contact stresses

Contact stress $p(r)$ as expressed by equation (4), becomes $P/(2\pi a^2)$ at the load center, where $r = 0$ and $-\infty$ at the load edge where $r = a$. Such kind of a load distribution can not be handled in any multilayer linear elastic software such as GAMES. In order for such a contract stress distribution to be analyzed, it is important to approximate its distribution by multiple uniformly distributed loads. The method that was used to approximate this distribution is explained below.

The load was subdivided into N different circular loads of different radii with same centers. The smallest radius is r_1 and the largest radius is $r_N = a$. Assuming every radius of each rigid plate circle to represent the total loading of P, we can obtain N domains comprised of the inner most circle of radius r_1 and rings formed by difference between adjacent circles

$r_i - r_{i-1}$ $(i \geq 2,)$.The value of r_i is determined by equality with the total loading. This means, total load in each domain is equal to $P_i = P/N$. This total load in each domain is assumed to be uniformly distributed and we can integrate equation (4) from r_{i-1} to r_i to obtain total load for the i^{th} domain. Considering that total load in each domain can be expressed as $P_i = P/N$, the following equation is obtained:

$$P\left(\sqrt{1-(r_{i-1}/a)^2} - \sqrt{1-(r_i/a)^2}\right) = \frac{P}{N} \quad \textbf{(6)}$$

Making use of the relationship between r_{i-1} and r_i shown in Equation (5), yields the following equation:

$$r_i = a\sqrt{1-\left(\sqrt{1-(r_{i-1}/a)^2} - 1/N\right)^2} \quad \textbf{(7)}$$

Replacing the contact stresses between r_{i-1} and r_i by uniformly distributed load p_i to obtain:

$$p_i = \frac{P}{N\pi(r_i^2 - r_{i-1}^2)} \quad \textbf{(8)}$$

Assume a load of magnitude $P = 1$ is acting on a circular rigid plate. In order for this load to be analyzed using GAMES software, it will be approximated as multiple concentric circular loads whose centers are on the loading plate. Furthermore, for the i^{th} load of radius r_i, responses (i.e. stresses and displacements) at point j can be represented by $\phi_j(r_i)$, and for $(i-1)^{th}$ load of radius r_{i-1}, responses can be represented by $\phi_j(r_{i-1})$. When a uniformly distributed load of magnitude $p = 1$ is acting on ring of width $(r_{i-1} \leq r < r_i)$, responses at point j can be obtained from the following equation:

$$\Delta\phi_{ji} = \phi_j(r_i) - \phi_j(r_{i-1}) \quad \textbf{(9)}$$

$\Delta\phi_{ji}$ represents i^{th} responses at point j when a uniformly distributed unit load is acting within a ring i. Superimposing responses on point j due to the effect of the loads acting on all the rings, the total response S_j may be obtained as follows:

$$S_j = \sum_{i=1}^{N} p_i \cdot \Delta\phi_{ji} = \sum_{i=1}^{N} \frac{P_i \cdot \Delta\phi_{ji}}{\pi(r_i^2 - r_{i-1}^2)} \quad \textbf{(10)}$$

p_i is the uniformly distributed load on ring i, while P_i is its total load.

Results from multiple uniformly distributed loading

When multiple uniformly distributed loads were analyzed, there was improvement in accuracy for bigger number of multiple loads but the down side of it was the increase in computational time. Three methods of load subdivisions were used where $N = 5, 10, 15$, and results from numerical analyses were compared with those obtained by Sneddon.

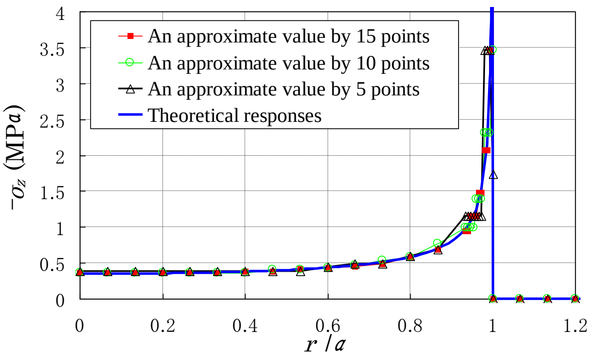


Figure 3 Theoretical and approximate contact stresses

($P = 49\text{kN}, E_1 = 100\,\text{MPa}, \nu = 0.35$)

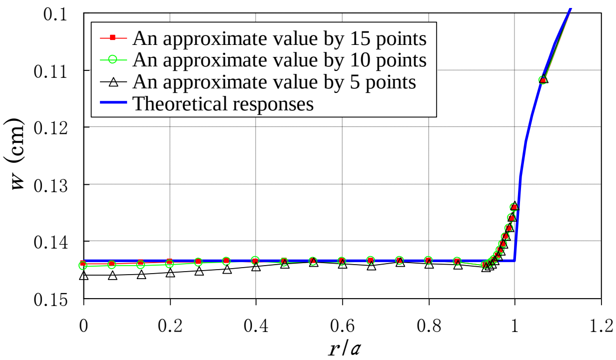
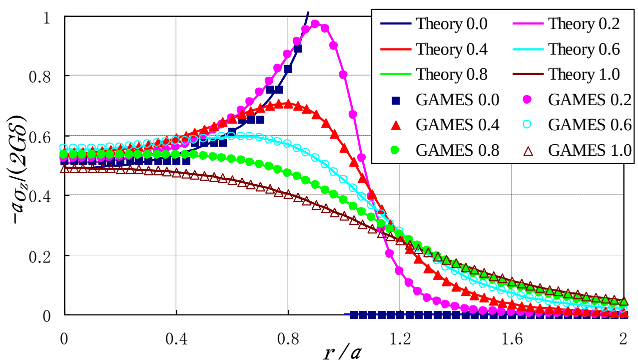


Figure 4 Subdivisions and surface deflections

($P = 49\text{kN}, E_1 = 100\,\text{MPa}, \nu = 0.35$)

Figure 3 is a comparison of contact stress distributions between theoretical analysis and approximated load distribution using the three different numbers of subdivisions. There is a significant difference between contact stresses near the load edge for the theoretical and approximate loads but as the number of multiple loads increases the contact stresses approach theoretical values. At the load edge, theoretical contact stress is infinite while approximate contact stress still maintains a finite value.

Figure 4 shows comparison of surface deflections for theoretical load as determined by equation (5) and surface deflections as computed by GAMES software for the three different N load subdivisions. This figure shows that as the number of load subdivisions N increases, results of surface deflections approach the theoretical values. However, it is not the same tendency for deflections near the load edge, where there is very little improvement as the number of load subdivisions increases. The reason may be attributed to the fact it is difficult for approximate loading to simulate theoretical loading near the edge, which approaches infinite. Deflection results by fewer load subdivisions, in this example 5 subdivisions, show poor accuracy near the load center while results from bigger number of subdivisions, that is $N = 10, 15$ show very good agreement with theoretical results. It is therefore recommended that at least 10 subdivisions are important to obtain reasonably good results.



＊ Legend value is z/a

Figure 5 Comparison of responses between approximate and theoretical loads

Figure 5 shows comparison of normal stresses (σ_z) at different depths from the surface (z/a) in order to confirm ability of 10 load subdivisions to approximate theoretical analysis. In this figure, dimensionless parameters of normal stresses (σ_z) divided by $2G\delta/a$

was plotted against r/a. Numbers in the legend of **Figure 5** indicate z/a ratios. In this figure, solid lines represent theoretical results while approximate results are represented by the marks as defined in the legend. The figure shows for surface results a difference of about 5% between theoretical and approximate loads and a very good agreement for points deeper than $z/a = 0.2$.

EXTENSION TO A TWO-LAYER STRUCTURE

Method for application evaluation

Equation (4) shows distribution of contact stresses resulting from a rigid plate loading acting on the surface of a semi-infinite medium and this distribution is not the same as for a two-layer structure. However, Saint-Venant's principle states that influence of load distribution is in the vicinity of the load application point and it is assumed that if the thickness of the surface layer is big enough, contact stress distribution for semi-infinite medium may be used in two-layer system without affecting computational accuracy.

Figure 6 shows a two-layer analytical model. Thickness of the first layer and the ratio of the elastic moduli between the first and second layer were varied and analyses performed in order to evaluate the limitation of the approximate method explained above. The accuracy of the analysis was evaluated in terms of deflections under the base of the rigid plate in order to see whether they remained constant. The root mean square error shown in equation (11) between mean deflection under the rigid plate and each deflection within the domain was used as evaluation criteria.

$$RMSE = \frac{1}{\overline{w}}\sqrt{\frac{1}{15}\sum_{j=1}^{15}\left(\overline{w} - w_j\right)^2} \qquad \textbf{(11)}$$

where $\overline{w}$ is the mean surface deflection and w_j is the deflection within $r = 0\text{\textasciitilde}14\text{cm}$. ($0 < r/a < 0.93$) at 1cm each ($r/a = 0.07$). Good agreement of displacements for all the points of interest will make $RMSE$ equal to zero. This will be an indication that approximate solution is equal to the theoretical solution. It is clear from **Figure 4** that irrespective of the number of load subdivisions, there is poor agreement between approximate solution and theoretical solution at the load edge, and hence results at the load edge position were not considered in equation (11).

The scope of application of the approximate loading was evaluated by using a two-layer system shown in **Figure 6** by following the procedure explained below:

1) Thickness of the first layer h_1 was varied from $h_1/a = 1$ to 3 in 0.2 steps resulting in 11 thickness variations.
2) For each value of h_1/a, E_1/E_2 was varied between 0.9~10 resulting in 11 variations.
3) h_1/a and E_1/E_2 variables resulted in 121 combinations, for which analyses were performed and example of the results is shown in **Figure 7**.
4) Using equation (11), each combination was evaluated and analytical errors $RMSE$ are given in **Table 1**.

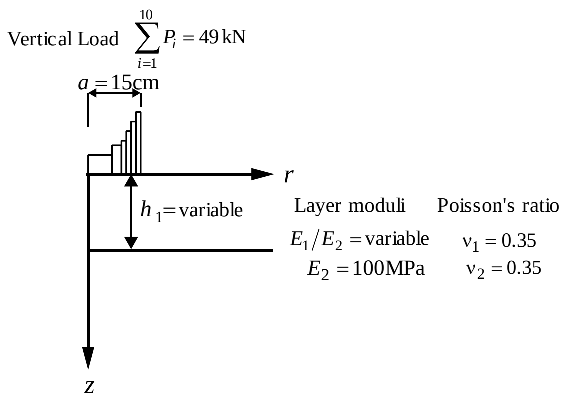
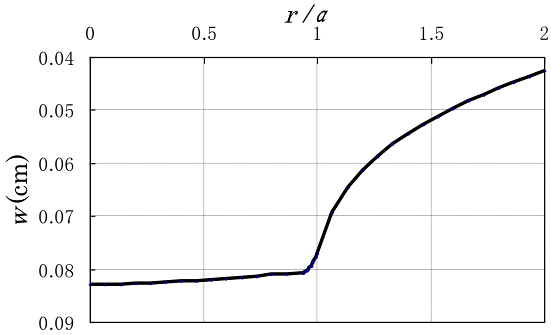


Figure 6 Two layer structure



$(h_1/a = 1.8,\ E_1/E_2 = 3)$

Figure 7 Surface deflections

Table 1 Least mean square error (LMSE) from Eq.(11)

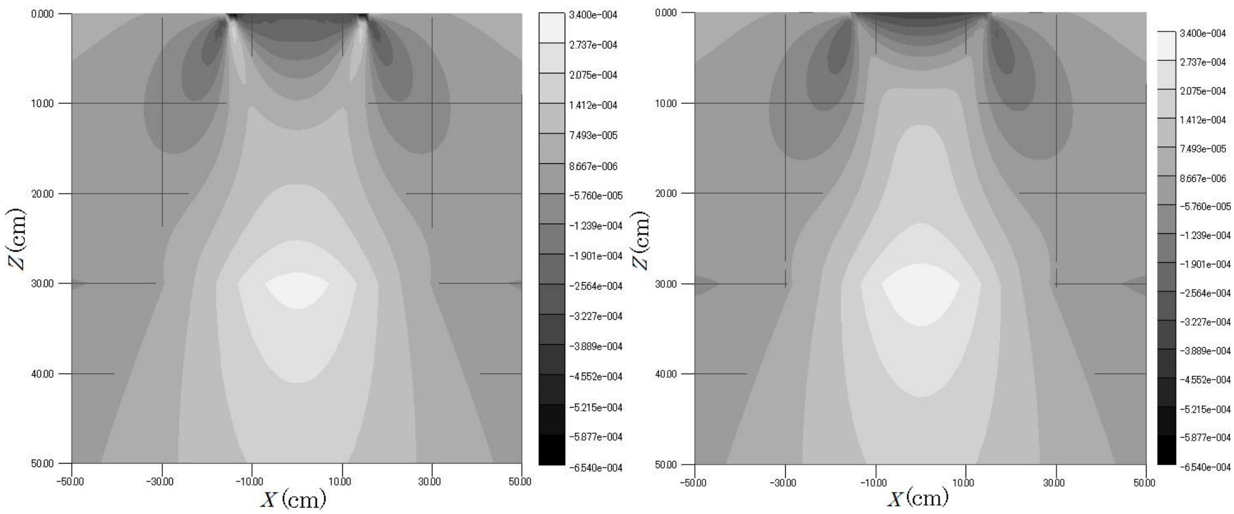
		h_1/a										
		1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
E_1/E_2	0.9	0.0035	0.0028	0.0024	0.0022	0.0021	0.0021	0.0021	0.0021	0.0021	0.0021	0.0021
	1	0.0021	0.0021	0.0021	0.0021	0.0021	0.0021	0.0021	0.0021	0.0021	0.0021	0.0021
	2	0.0139	0.0139	0.0108	0.0085	0.0069	0.0056	0.0047	0.0040	0.0035	0.0032	0.0029
	3	0.0238	0.0184	0.0144	0.0113	0.0091	0.0074	0.0061	0.0052	0.0044	0.0039	0.0034
	4	0.0262	0.0203	0.0159	0.0126	0.0101	0.0082	0.0068	0.0057	0.0049	0.0042	0.0037
	5	0.0271	0.0210	0.0165	0.0131	0.0105	0.0086	0.0071	0.0060	0.0051	0.0044	0.0039
	6	0.0274	0.0213	0.0167	0.0133	0.0107	0.0088	0.0073	0.0061	0.0052	0.0045	0.0040
	7	0.0274	0.0213	0.0167	0.0133	0.0107	0.0088	0.0073	0.0062	0.0053	0.0045	0.0040
	8	0.0272	0.0211	0.0166	0.0132	0.0107	0.0088	0.0073	0.0062	0.0053	0.0045	0.0040
	9	0.0269	0.0209	0.0164	0.0131	0.0106	0.0087	0.0073	0.0061	0.0052	0.0045	0.0040
	10	0.0266	0.0206	0.0162	0.0129	0.0105	0.0086	0.0072	0.0061	0.0052	0.0045	0.0039

Table 1 shows errors ($RMSE$) for each combination considered. With regards to the ratios of layer moduli and irrespective of the h_1/a value, "ratio 7" gives the largest $RMSE$ value. Smaller ratios of layer moduli, where the two-layer system is close to semi-infinite medium, result in smaller errors in the surface deflections. Practical application of the approximate method should be for all combinations with $h_1/a \geq 2$ and $RMSE$ less than 1%. All the errors falling outside this criterion have been grayed out in **Table 1**.

Figure 7 shows surface deflection w for the acceptable region of application with the highest error $(h_1/a = 1.8,\ E_1/E_2 = 3)$. Surface deflection at the center of the load is 0.083 cm, while at the load edge where ($r/a = 0.933$) the surface deflection is 0.081 cm, showing considerably smaller error.

Rigid plate versus uniformly distributed loads

An example for the above case with largest $RMSE$ was assumed. Thickness of the first layer was $h_1 = 30\text{cm}$ and layer modulus was $E_1 = 700\,\text{MPa}$. Elastic modulus of the second layer was $E_2 = 100\,\text{MPa}$. Radius of the load was $a = 15\text{cm}$ and analyses for a 49kN was performed considering a rigid plate load and uniformly distributed load. Results were compiled and presented in Cartesian coordinates where **Figure 8(a)** shows contour plot of ε_x due to rigid plate load and **Figure 8(b)** is for uniformly distributed load. At the boundary between the first and second layer where $z = 30\text{cm}$ and just directly under the load, both



(a) Rigid plate load (b) Uniformly distributed load

Figure 8 Contour plots for ε_x in X-Z plane

loads result in tensile strains but results from the rigid plate loading cover a relatively smaller area. For the case of uniformly distributed load, there are high strains, which are transmitted downwards while for the rigid plate loading strains are transmitted from the load edge downwards.

CONCLUSIONS

An approximate method was developed whereby GAMES software, which is capable of analyzing effect of uniformly distributed load on a multilayered linear elastic system, was utilized to analyse a multilayered linear elastic system under the action of rigid plate loading. Results obtained are summarized as follows:

1) With an exception of rigid plate edge, very accurate results were obtained when load subdivision is greater or equal to 10.
2) It was possible to accurately evaluate a multilayered linear elastic system using rigid approximate contact stresses for a semi-infinite medium.
3) There were very high strains developed at the layer interface between the first and second layers for the case of uniformly distributed load. However, because of better load dispersion from the rigid plate edge downward, relatively low strains were developed at the layer interface right below the load center.
4) For $h_1/a > 2$, contact stresses for rigid plate loading on a semi-infinite medium may be used to analyze a multilayered linear elastic system irrespective of the magnitude of layer moduli ratios.

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