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Inclusions and inhomogeneities under stress†

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ABSTRACT

Some general theorems, new and old, concerning the behaviour of elastic inclusions and inhomogeneities in bodies without or with external stress, are assembled. The principal new result is that arbitrary external tractions cannot influence the shape of an inclusion, because the elastic field of the external tractions passes unperturbed through the inclusion, and the work done by the external forces is independent of the shape of the inclusion. The terms in the energy quadratic in the external tractions are unaffected by the presence of the inclusion, and by Colonnetti's theorem the total elastic energy contains no cross terms between internal and external stresses.

§ 1. INTRODUCTION

Mott was probably the first to attempt a quantitative estimate of the influence of dispersed particles of a second phase on the plastic properties of a crystal (Mott and Nabarro 1940). Since then, many types of problem have been considered. Many studies are concerned with coherent precipitates, in which each lattice point retains its identity and the structure remains topologically a single crystal, although elastically distorted. Others (for example Nabarro (1940)) treated the special case in which the total number of atoms within the boundary of the precipitate is conserved, but the atoms are free to fill their allotted space in the way which minimizes the elastic energy. The precipitate is effectively fluid. In dislocation theory, one is sometimes concerned with particles which are free from elastic stress but are not easily penetrated by dislocations having the Burgers vector of a dislocation in the matrix.

Where elastic stresses are present, it is useful to distinguish between an *inclusion*, which has the same elastic constants as those of the matrix, and an *inhomogeneity*, which has different elastic constants. In isotropic elasticity there is an intermediate class, in which the shear moduli are equal but the bulk moduli differ. We shall often assume that the inclusion or inhomogeneity is derived from the matrix by a *transformation* which involves a homogeneous (although not necessarily isotropic) spontaneous elastic strain.

A further problem of interest is to understand the way in which an external stress influences the equilibrium shapes of precipitate particles. This problem has become important in the application of so-called superalloys in aircraft turbine engines, where a two-phase alloy is subjected to very high temperatures and stresses, and the particles of precipitate undergo a change of shape known as rafting (for example

† Dedicated to Sir Nevill Mott on his ninetieth birthday.

Nabarro (1995) and Nabarro, Cress and Kotschy (1995)). The quantitative solutions of problems of this kind often depend on numerical computations, which may (Socrate and Parks 1993) be based on Eshelby's (1969, 1975) energy–momentum tensor. All these studies take place within the framework of linear elasticity, sometimes coupled with the assumption of isotropic plasticity in one of the phases, and the interfacial surface energy is neglected. Numerical computations often give little insight into the underlying physical principles, and it seems useful to assemble some general results which bear on problems of this type. They cannot provide solutions, but provide useful checks on the validity of solutions obtained by other means.

§ 2. Transformations in bodies free from external stress

A theorem in isotropic elasticity which J. W. Cahn informs me is originally due to Bitter (1931), and which has repeatedly been rediscovered independently (Goodier 1937, M. M. Crum 1940, private communication (see Nabarro (1940)), Robinson 1951) states that, if a homogeneous dilatation of a prescribed magnitude occurs in a volume V of an infinite isotropic medium, and the material after the transformation has the same shear modulus as before the transformation (but not necessarily the same bulk modulus), then the total elastic energy is proportional to V but independent of the shape of the transformed body.

This remarkable result has a very simple physical interpretation. Consider a spherical inhomogeneity. In isotropic elasticity, the strain field inside the inhomogeneity is purely dilatational, while that outside is free from dilation and governed by the shear modulus alone. Now consider two such inhomogeneities, and suppose that the shear modulus of the inhomogeneities is the same as that of the matrix. Then the elastic field of the first inhomogeneity is not affected by the presence of the second, and so (Eshelby 1961) the work done in creating the second inhomogeneity is not altered by the stress field of the first, and there is no elastic interaction energy between two spherical inhomogeneities which have the same shear modulus. One region of inhomogeneity may be moved with respect to the other without changing the total elastic energy. The prescribed volume V may be filled to any desired accuracy with a set of spherical particles of various sizes. Since no work is done in assembling these particles from a dilute distribution, the elastic energy is independent of the shape of the final assembly.

This line of thought leads at once to an additional result. Suppose that the volume V lies, not in an infinite medium, but in the deep interior of a body large compared with the inhomogeneity. Then the change in the external volume of the body produced by the dilatational transformation is independent of the shape of the transformed volume, because it is simply the sum of the changes in volume produced by the individual spherical particles.

In isotropic elasticity, if the precipitate is effectively fluid, the total elastic energy tends to zero if the prescribed volume V takes the form of a thin disc (Nabarro 1940).

In general anisotropic linear elasticity, the elastic energy of a coherent inclusion with the same elastic constants as those of the matrix is a minimum if the inclusion is rolled out into an infinitely thin plate whose normal is determined by the elastic constants (Khachaturyan 1983).

Because of crystal symmetry, this normal is usually not uniquely determined. What is determined is a set of crystallographically equivalent normals. In the practically interesting case of a family of precipitate shapes having orthorhombic symmetry with axes along the cubic axes of a matrix with cubic symmetry, it is easy to show (Nabarro *et al.* 1995) that the equiaxed form represents a configuration of equilibrium, not necessarily stable.

We now consider a transformation occurring in a region of a finite body which is not necessarily small in comparison with the body as a whole. Then, if the transformation consists of a homogeneous strain without change in elastic constants, the volume of the body containing the transformed region is the sum of the volumes of the transformed region and of the matrix when disassembled. To show this, we consider the application of a small test pressure p. For ease of drawing, we take the transformation to involve a shrinkage of the inclusion (figure 1). There are three regions: region 0 outside the body, region 1 which does not undergo transformation, and region 2 which undergoes transformation. The interfaces are S_{10} and S_{21}.

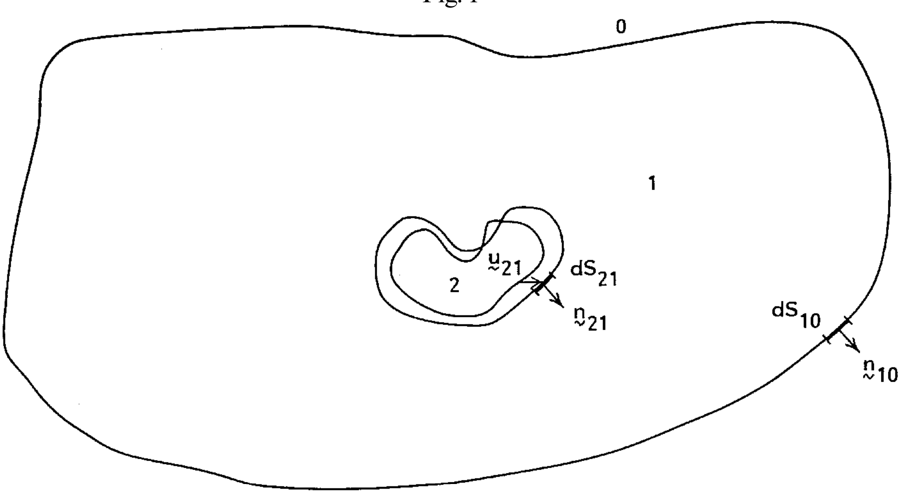
We consider the following cycle.

(1) The unchanged portion 1 and the transformed portion 2 are assembled by joining corresponding points on the inner and outer surfaces of S_{21}, separated by $\mathbf{u}_{21}$ after the expansion.
(2) A small test pressure $-p$ is applied.
(3) The body is disassembled in the presence of $-p$.
(4) The pressure $-p$ is removed, and the system has returned to its original state. Let the volumes of regions 1 and 2 before the expansion be V_1 and V_2. Let the volume increase of V_2 during free expansion be δV_2, and let the volume of the coherently constrained body after the expansion be $V_1 + V_2 + \Delta V$.

Let the work done by external forces and stored in the body when regions 1 and 2 are assembled be W_a, and let the bulk modulus of the material be K.

Then the increases in the stored energy of the system in the four steps of the cycle are

Fig. 1



A finite body with boundary S_{10} contains an inclusion with boundary S_{21}. The inclusion undergoes a homogeneous transformation strain which, if continuity across S_{21} were broken, would cause a relative displacement by $\mathbf{u}_{21}$ of corresponding points on opposite sides of the interface.

(1) W_a,
(2) $(p^2/2K)(V_1 + V_2 + \Delta V)$ (Colonnetti's theorem (§3) shows that there are no cross terms),
(3) $-W_a - p\Delta V + p\delta V_2 + O[(p^2/2K)\delta V] + O[(p^2/2K)\delta V_2]$ (the term W_a is unaltered because the applied pressure creates the same distortion in the inclusion and in the cavity into which it is fitted), and
(4) $-(p^2/2K)(V_1 + V_2)$.

The sum of these energies must vanish. Since $p \ll K$, this requires that

$$\Delta V = \delta V_2 . \tag{1}$$

This result may easily be verified for the case of a sphere in a concentric spherical shell by using the results of Love (1920) with $p_0 = 0$.

§3. Transformations in bodies under homogeneous external stress

The most powerful theorem is due to Colonnetti (1915). It applies to any linear elastic system, with no assumptions as to isotropy or homogeneity. The total elastic energy of a linear system subjected to internal and external stresses has no cross-terms between the internal and the external stresses. The proof is trivial. Suppose that, in the presence of the final external stresses, the displacement of any surface element dA is $\mathbf{u}(\mathrm{d}A)$, while the traction across this element is $\mathbf{t}(\mathrm{d}A)$. Suppose now that these tractions are applied gradually and proportionately to all elements, so that during the process the displacements are $\lambda\mathbf{u}(\mathrm{d}A)$ and the tractions $\lambda\mathbf{t}(\mathrm{d}A)$. For the body containing internal stresses but free from external stresses, $\lambda = 0$; for the body subject to internal and external stresses, $\lambda = 1$. Let the energy of the body free from external stresses be U_i. Then the energy of the body under both internal and external stresses is

$$\begin{aligned} U &= U_i + \int_0^1 \int \lambda\mathbf{t}(\mathrm{d}A)\cdot\mathbf{u}(\mathrm{d}A)\,\mathrm{d}A\,\mathrm{d}\lambda \\ &= U_i + \tfrac{1}{2}\int \mathbf{t}(\mathrm{d}A)\cdot\mathbf{u}(\mathrm{d}A)\,\mathrm{d}A, \end{aligned} \tag{2}$$

and there are no cross terms between the internal stresses and the tractions $\mathbf{t}$.

It follows from §2 that, if a transformation involves a homogeneous strain without change in elastic constants, the volume of the body containing the transformed region is independent of the shape of the transformed region. Thus an external hydrostatic pressure does no work if the shape of the transformed region changes at constant volume, and hydrostatic pressure cannot induce such a change in shape.

Suppose now that the hydrostatic pressure mentioned above is replaced by an arbitrary externally applied homogeneous stress. Then, if the transformation is a homogeneous strain without change of elastic constants, an argument similar to that in §2 shows that the work done by the external stress during the transformation is the same as the work which would be done if the transformation occurred in the isolated inclusion under this stress. This work is independent of the shape of the inclusion, and therefore the arbitrary stress cannot induce a change in the shape of the inclusion.

The result is really intuitive, because the external stress field passes unperturbed through the inclusion. The terms in the energy quadratic in the external stress are unaffected by the presence of the inclusion while, by Colonnetti's theorem, there are no cross-terms between the internal and the external stresses. This interpretation allows the previous result to be generalized; apart from its elastic response, the equilibrium shape of an elastic inclusion is not influenced by an arbitrary set of external tractions.

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