ICSV13 - Vienna
The Thirteenth International Congress on Sound and Vibration
Vienna, Austria, July 2-6, 2006

RESONANT VIBRATIONS AND ACOUSTIC RADIATION OF ROTATING SPHERICAL STRUCTURES

Michael Shatalov*[1], Igor Fedotov[2], and Stephan Joubert[2]

[1] Sensor Science and Technology (SST) of CSIR Material Science and Manufacturing, P.O. Box 395, Pretoria 0001, CSIR, South Africa and
Department of Mathematical Technology, P.B.X680, Pretoria 0001 FIN-40014, Tshwane University of Technology, South Africa.
[2] Department of Mathematical Technology, P.B.X680, Pretoria 0001, FIN-40014 Tshwane University of Technology, South Africa
mshatlov@csir.co.za (e-mail address of lead author)

Abstract

The generating equations of the problem are considered in terms of a system of three dimensional equations of linear elasticity considered in the spherical coordinates. It is known that in this case the exact solution of the problem could be obtained in the spherical Bessel, associate Legendre and trigonometric functions. The spherical coordinates are introduced so that a constant vector of the inertial angular rate passes through the pole of the coordinates. It is supposed that the angular rate of the inertial rotation is much smaller than a minimal circular frequency of elastic vibrations of the structure and hence, it is possible to neglect the centrifugal forces. It is shown that the elastic waves of the structure are partially involved into rotation (precession) with respect to the inertial space with scale factors depending on nature of elastic modes and their numbers. Corresponding scales factors, or Bryan's factors of the vibrating mode's precession are calculated depending on nature of the modes, spheroidal or torsional and their numbers. Bryan's factors of radiated spherical body are calculated and compared with corresponding factors of a free body.

INTRODUCTION: STATEMENT OF THE PROBLEM

Let us consider an isotropic and elastic solid sphere of radius $r = a$ (Fig. 1). It is supposed that the sphere is surrounded by an acoustic medium, which will be considered as an ideal non-viscous fluid. Suppose that the sphere is subjected to an inertial rotation with small constant angular rate Ω coinciding with axis Oz. Terms, proportional to the square of Ω will be neglected, i.e. it is supposed that $O\left(\Omega^2\right) \approx 0$.

We introduce the following systems of coordinates:

Eds.: J. Eberhardsteiner, H.A. Mang, H. Waubke

$Oxyz$ – rotates in an inertial reference frame with the sphere about Oz - axis with angular rate Ω;
$Ox_1y_1z_1$ - rotated over $Oxyz$ at angle φ - over Oz;
$Ox_2y_2z_2$ - rotated over $Ox_1y_1z_1$ at angle θ - over Oy_1.

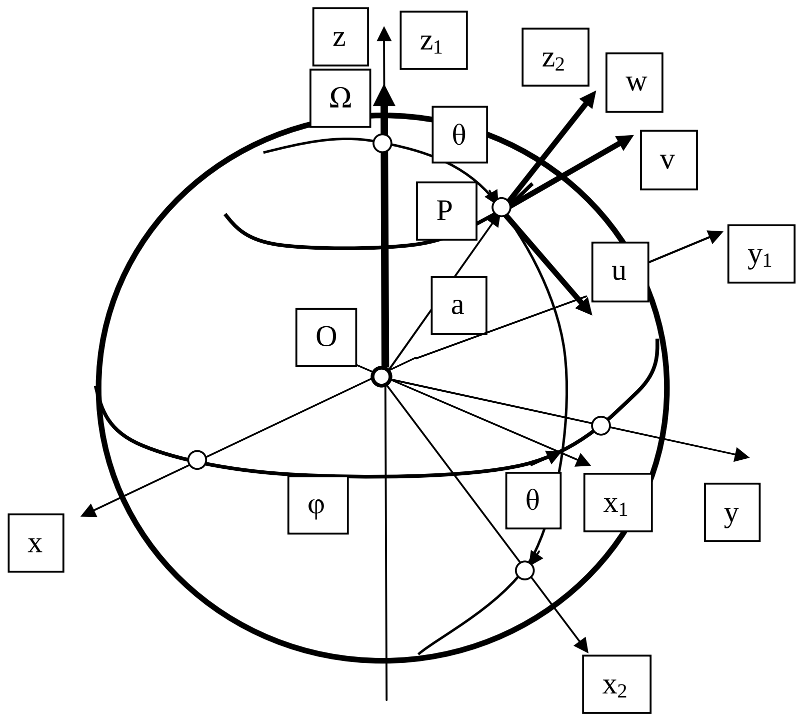

Figure 1 – Coordinate systems for spherical body

GENERATING SOLUTION FOR A NON-ROTATING SPHERE

Generating equations of motion of motion of the spherical body ($\Omega = 0$) are[1]:

$$\begin{cases} \dfrac{\partial\ \sigma_{rr}}{\partial r}+\dfrac{1}{r}\dfrac{\partial\ \sigma_{r\theta}}{\partial\theta}+\dfrac{1}{r\sin\theta}\dfrac{\partial\ \sigma_{r\varphi}}{\partial\varphi}+\dfrac{2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{\varphi\varphi}-\cot\theta\cdot\sigma_{r\theta}}{r}=\rho\dfrac{\partial^2 w}{\partial t^2} \\ \dfrac{1}{r}\dfrac{\partial\ \sigma_{\theta\theta}}{\partial\theta}+\dfrac{\partial\ \sigma_{r\theta}}{\partial r}+\dfrac{1}{r\sin\theta}\dfrac{\partial\ \sigma_{\theta\varphi}}{\partial\varphi}+\dfrac{3\sigma_{r\theta}+\cot\theta\cdot\left(\sigma_{\theta\theta}-\sigma_{\varphi\varphi}\right)}{r}=\rho\dfrac{\partial^2 u}{\partial t^2} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial\ \sigma_{\varphi\varphi}}{\partial\varphi}+\dfrac{\partial\ \sigma_{r\varphi}}{\partial r}+\dfrac{1}{r}\dfrac{\partial\ \sigma_{\theta\varphi}}{\partial\theta}+\dfrac{3\sigma_{r\varphi}+2\cot\theta\cdot\sigma_{\theta\varphi}}{r}=\rho\dfrac{\partial^2 v}{\partial t^2} \end{cases} \quad (1)$$

where the stresses:

$$\sigma_{rr} = \lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\varphi\varphi}\right)+2\mu\varepsilon_{rr}; \qquad \sigma_{\theta\theta} = \lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\varphi\varphi}\right)+2\mu\varepsilon_{\theta\theta};$$
$$\sigma_{\varphi\varphi} = \lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\varphi\varphi}\right)+2\mu\varepsilon_{\varphi\varphi}; \quad \sigma_{r\theta}=\mu\varepsilon_{r\theta}; \quad \sigma_{r\varphi}=\mu\varepsilon_{r\varphi}; \quad \sigma_{\theta\varphi}=\mu\varepsilon_{\theta\varphi} \tag{2}$$

and strains are:

$$\varepsilon_{rr}=w'_r; \qquad \varepsilon_{\theta\theta}=\frac{1}{r}\left(u'_\theta+w\right); \qquad \varepsilon_{\varphi\varphi}=\frac{1}{r}\left(\cot\theta\cdot u+\frac{1}{\sin\theta}v'_\varphi+w\right);$$
$$\varepsilon_{r\theta}=u'_r+\frac{1}{r}\left(-u+w'_\theta\right); \quad \varepsilon_{r\varphi}=v'_r+\frac{1}{r}\left(-v+\frac{1}{\sin\theta}w'_\varphi\right); \quad \varepsilon_{\theta\varphi}=\frac{1}{r}\left(\frac{1}{\sin\theta}u'_\varphi-\cot\theta\cdot v+v'_\theta\right) \tag{3}$$

By means of change of variables $(u,v,w)\to(\Phi,\Psi,\mathrm{X})$:

$$w=\Phi'_r+r\cdot\left[\left(\mathrm{X}''_{rr}+\frac{2}{r}\mathrm{X}'_r\right)-\nabla^2\mathrm{X}\right]; \quad u=\left[\mathrm{X}'_r+\frac{1}{r}(\Phi+\mathrm{X})\right]'_\theta+\frac{1}{a\sin\theta}\Psi'_\varphi; \quad v=\frac{1}{\sin\theta}\cdot\left[\mathrm{X}'_r+\frac{1}{r}(\Phi+\mathrm{X})\right]'_\varphi-\frac{1}{a}\Psi'_\theta \tag{4}$$

where $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$ - Laplacian in the spherical coordinates, the variables are separated:

$$(\lambda+2\mu)\cdot\nabla^2\Phi=\rho\ddot{\Phi}; \qquad \mu\cdot\nabla^2\Psi=\rho\ddot{\Psi}; \qquad \mu\cdot\nabla^2\mathrm{X}=\rho\ddot{\mathrm{X}} \tag{5}$$

Considering a steady-state process $\left(\frac{d}{dt}\to i\,\omega, \quad \frac{d^2}{dt}\to-\omega^2\right)$ the solutions of equations (5) for the case of a solid sphere could be represented as a sum of *mn* – spherical harmonics:

$$\begin{cases}\Phi_{mn}(r,\theta,\varphi)=A_{mn}\cdot j_n(k_1 r)\cdot P_n^m(\cos\theta)\cdot\cos(m\varphi)\\ \mathrm{X}_{mn}(r,\theta,\varphi)=B_{mn}\cdot j_n(k_2 r)\cdot P_n^m(\cos\theta)\cdot\cos(m\varphi)\\ \Psi_{mn}(r,\theta,\varphi)=D_{mn}\cdot j_n(k_2 r)\cdot P_n^m(\cos\theta)\cdot\sin(m\varphi)\end{cases} \tag{6}$$

where the wave numbers $k_1=k_1(\omega)=\omega/c_1$; $k_2=k_2(\omega)=\omega/c_2$ and $c_1=\sqrt{(\lambda+2\mu)/\rho}$; $c_2=\sqrt{\mu/\rho}$ - speeds of extensional and inextensional waves propagation.

Due to absence of radial and tangential stresses on the spherical surface ($r=a$) the boundary conditions are:

$$\left[\sigma_{rr}\right]_{r=a}=\left[\sigma_{r\theta}\right]_{r=a}=\left[\sigma_{r\varphi}\right]_{r=a}=0 \tag{7}$$

It is possible to show that could be satisfied for two different modes of vibration:

- *Spheroidal modes:*

$$\begin{cases} \left\{\left[\Phi''_{rr} - \frac{k_2^2(\omega)}{2}\cdot\frac{\lambda}{\lambda+2\mu}\cdot\Phi\right] + \frac{2}{r}\cdot\left[k_2^2(\omega)\cdot(r\mathrm{X}'_r+\mathrm{X}) + (r\mathrm{X}'_r+\mathrm{X})''_{rr}\right]\right\}_{r=a} = 0 \\ \left\{\frac{1}{r}\cdot\left[\Phi'_r - \frac{1}{r}\cdot\Phi\right] + \left[\mathrm{X}''_{rr} + \frac{1}{r}\cdot\mathrm{X}'_r + \left(\frac{k_2^2(\omega)}{2} - \frac{1}{r^2}\right)\cdot\mathrm{X}\right]\right\}_{r=a} = 0 \end{cases} \tag{8}$$

- *Torsional modes:*

$$\left[\Psi'_r - \frac{1}{r}\cdot\Psi\right]_{r=a} = 0 \tag{9}$$

After solution of the characteristic equations (8) – (9) we define eigenvalues $\tilde{\omega}$. Substituting them into (6) and (4) the eigenfunctions $U_{mn} = U_{mn}(r,\theta)$, $V_{mn} = V_{mn}(r,\theta)$, $W_{mn} = W_{mn}(r,\theta)$ are obtained.

PRECESSING WAVES IN VIBRATING AND ROTATING SPHERE

Angular rate Ω in projections on $Ox_2y_2z_2$ (Fig-1) is $\vec{\Omega} = [-\Omega\sin\theta \quad 0 \quad \Omega\cos\theta]^T$. Radius-vector of a deflected point *P* in projections on these axes $\vec{r} = [u \quad v \quad r+w]^T$. According to the Euler's formula, the absolute linear velocity of point *P*:

$$\vec{V} = \dot{\vec{r}} + \vec{\Omega}\times\vec{r} = \begin{bmatrix} \dot{u} - \Omega v\cos\theta \\ \dot{v} + \Omega\left[u\cos\theta + (r+w)\sin\theta\right] \\ \dot{w} - \Omega v\sin\theta \end{bmatrix} \tag{10}$$

Suppose that a solution of the equations of motion is obtained for the *mn* –mode as follows:

$$\begin{aligned} u_{mn} &= U_{mn}(r,\theta)\cdot\left[C_{mn}(t)\cos m\varphi + S_{mn}(t)\sin m\varphi\right] \\ v_{mn} &= V_{mn}(r,\theta)\cdot\left[S_{mn}(t)\cos m\varphi - C_{mn}(t)\sin m\varphi\right] \\ w_{mn} &= W_{mn}(r,\theta)\cdot\left[C_{mn}(t)\cos m\varphi + S_{mn}(t)\sin m\varphi\right] \end{aligned} \tag{11}$$

where $C_{mn}(t), S_{mn}(t)$ - time dependent functions; $U_{mn}(r,\theta), V_{mn}(r,\theta), W_{mn}(r,\theta)$ - eigenfunctions.

Kinetic energy of the solid sphere (for the sake of brevity we omit mn - indices):

$$T = \frac{\rho}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left\|\vec{V}\right\|^2 r^2\sin\theta\, dr\, d\theta\, d\varphi = T\left(\dot{C},\dot{S},C,S\right) \approx \frac{1}{2}I_0\cdot\left(\dot{C}^2+\dot{S}^2\right) + \Omega\cdot I_1\cdot\left(C\dot{S} - \dot{C}S\right) \tag{12}$$

where the terms $O\left(\Omega^2\right)$ are neglected and

$$I_0 = \rho \cdot \int_0^{\pi}\int_0^{a}\left[U^2(r,\theta)+V^2(r,\theta)+W^2(r,\theta)\right]\cdot r^2 \cdot \sin\theta \, dr\, d\theta;$$
$$I_1 = 2\rho \cdot \int_0^{\pi}\int_0^{a}\left[\left(U(r,\theta)\cdot\cos\theta + W(r,\theta)\cdot\sin\theta\right)\cdot V(r,\theta)\right]\cdot r^2 \cdot \sin\theta \, dr\, d\theta \qquad (13)$$

Substituting (11) in (2)-(3) we obtain the following expression for potential energy:

$$P(C,S) = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left(\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{\varphi\varphi}\varepsilon_{\varphi\varphi} + \sigma_{r\theta}\varepsilon_{r\theta} + \sigma_{r\varphi}\varepsilon_{r\varphi} + \sigma_{\theta\varphi}\varepsilon_{\theta\varphi}\right)\cdot r^2 \sin\theta \, dr\, d\theta\, d\varphi = \frac{1}{2} I_2\left(C^2+S^2\right) \qquad (14)$$

where σ_{rr} , ε_{rr}, ... – see (2)-(3), and $I_2 = I_2(m,n)$. The Lagrangian of the mn-vibrating pattern is:

$$L = L\left(\dot{C},\dot{S},C,S\right) = T\left(\dot{C},\dot{S},C,S\right) - P(C,S) \approx \frac{1}{2} I_0 \cdot \left(\dot{C}^2+\dot{S}^2\right) + \Omega\cdot I_1\cdot\left(C\dot{S}-\dot{C}S\right) - \frac{1}{2} I_2\left(C^2+S^2\right) \qquad (15)$$

Equations of motion are:

$$\begin{cases} \ddot{C} - 2\eta\,\Omega\,\dot{S} + \omega^2 C = 0 \\ \ddot{S} + 2\eta\,\Omega\,\dot{C} + \omega^2 S = 0 \end{cases} \qquad (16)$$

where $\omega^2 = \omega_{mn}{}^2 = I_2 / I_0$. The Bryan's factor $\eta = \eta_{mn}$ is[2]:

$$\eta = \frac{I_1}{I_0} \qquad (17)$$

It is simply to prove that $0 \le |\eta| \le 1$. Furthermore, it is possible to prove that η is an geometric invariant (it does not depend on radius of the sphere $r = a$) as well as mass and stiffness invariant (it does not does not depend on mass density ρ and modulus of elasticity E of the elastic material). The Bryan's factor depends on Poisson's ratio ν (see example).

Let us consider the effect of the skew-symmetric gyroscopic forces $-2\eta\Omega\dot{C},\ \ 2\eta\Omega\dot{S}$ on dynamics of vibrating pattern[3]. We multiply the second equation (16) by $i\left(i^2=-1\right)$, add with the first equation and introducing a new complex variable $Z = C + iS$ ($iZ = -S + iC$) obtain the following equation:

$$\ddot{Z} + 2i\eta\,\Omega\,\dot{Z} + \omega^2 Z = 0 \qquad (18)$$

Let us change variable $Z \to Y$: $Z(t) = Y(t)\cdot e^{i\alpha t}$, where $\alpha = const$ will be defined later. In this case $\dot{Z} = \left(\dot{Y} + i\alpha Y\right)\cdot e^{i\alpha t}$; $\ddot{Z} = \left(\ddot{Y} + 2i\alpha\dot{Y} - \alpha^2 Y\right)\cdot e^{i\alpha t}$. Substituting these expressions in equation (16) we obtain:

$$\ddot{Y} + 2i\left(\alpha + \eta\,\Omega\right)\dot{Y} + \left(\omega^2 - \alpha^2 - 2\,\alpha\eta\,\Omega\right)Y = 0 \qquad (19)$$

It is obvious that term $\dot{Y}$ could be eliminated if we choose $\alpha = -\eta\,\Omega$. In this case $\omega^2 - \alpha^2 - 2\,\alpha\eta\,\Omega = \omega^2 + \eta^2\Omega^2 \approx \omega^2$ because we neglect terms $O\left(\Omega^2\right)$. In this case equation (19) is simplified to the equation of a harmonic oscillator $\ddot{Y} + \omega^2 Y \approx 0$. It means that using the transformation $Z(t) = Y(t)\cdot e^{-i\eta\Omega t}$ we fix the vibrating pattern in the reference frame, which rotates with angular rate $\Omega = -\eta\,\Omega$ relative to the rotating reference frame $Oxyz$. In the immovable reference frame $O\xi\eta\varsigma$ we observe the vibrating pattern rotation with angular rate $\Omega = (1-\eta)\,\Omega$.

Hence, we defined a new object, the so-called *precessing wave*. The effect of precession is defined by the abovementioned gyroscopic forces, proportional to the first power of inertial angular rate Ω. It is necessary to stress that the Bryan's factor $\eta = \eta_{mn}$ substantially depends on particular *mn* – eigenfunctions.

ROTATING AND VIBRATING SPHERE IN ACOUSTICAL MEDIUM

Due to presence of a surrounding acoustical medium the boundary conditions (7) are rewritten as follows:

$$[\sigma_{rr}]_{r=a} + \left[p^{(m)}\right]_{r=a} = 0;\quad [w]_{r=a} - \left[w^{(m)}\right]_{r=a} = 0;\quad [\sigma_{r\theta}]_{r=a} = \left[\sigma_{r\varphi}\right]_{r=a} = 0 \qquad (20)$$

The first expression means the equality of radial stress of the sphere to the external pressure in the acoustical medium and the second – equality of radial displacements of the sphere and the medium at $r = a$. mn – component of pressure in the acoustic medium is ($h_n^{(2)}(kr) = j_n(kr) - i\cdot y_n(kr)$ - Hankel spherical function, $c^{(m)}$ - speed of sound in the acoustical medium) $p_{mn} = p_{mn}(r,\theta,\varphi,\omega) = h_n^{(2)}\left(\omega/c^{(m)}\, r\right)P_n^m(\cos\theta)\left[P_{mn}^{(c)}\cos(m\varphi) + P_{mn}^{(s)}\sin(m\varphi)\right]$. Radial displacement of the medium is $w^{(m)} = p^{(m)}/\left(\rho^{(m)}\omega^2\right)$. Tangential components of the stress are zero because the medium's viscosity is neglected.

CALCULATION OF EIGENVALUES AND BRYAN'S FACTORS

Let us consider an example of a sphere of radius $a = 0.5\,m$ made from an aluminium alloy with modulus of elasticity $E = 7\cdot 10^{10}\ N/m^2$, Poisson's ration $\nu = 0.33$ and mass density $\rho = 2.7 \cdot 10^3 \, {kg}/{m^3}$. Calculated real values of eigenvalues in Hz of spheroidal modes are given in the Table - 1 for the cases of free outer surface and acoustically loaded surface for $n=2, m=2;\quad n=3, m=2;\quad n=3, m=3$. Parameters of the acoustic medium are: sound speed - $c^{(m)} = 1500 \, {m}/{s}$ and mass density - $\rho^{(m)} = 1000 \, {kg}/{m^3}$. Corresponding values of the Bryan's factors of the spheroidal modes are also given in the Table - 1.

Table – 1. Eigenvalues and corresponding Bryan's factors

m=2	Eigenvalues (free boundary, Hz)	Bryan's factor (free boundary)	Eigenvalues (**Re**, acoustic medium, Hz)	Bryan's factor (acoustic medium)
n=2	2633	0.921	2567	0.969
	5056	0.137	5050	0.403
	8563	0.300	6616	0.333
			8190	0.350
	10848	0.270	9728	0.329
n=3	3924	0.515	2296	0.608
	6654	0.127	5660	0.175
			6654	0.174
	9910	0.136	7252	0.173
			8824	0.182
m=3	Eigenvalues (free boundary, Hz)	Bryan's factor (free boundary)	Eigenvalues (**Re**, acoustic medium, Hz)	Bryan's factor (acoustic medium)
n=3	3924	0.634	2296	0.708
	6654	0.000	5660	0.149
			6654	0.151
	9910	0.073	7252	0.103
			8824	0.110

It follows from this Table that the precessing waves of the spheroidal modes move in the direction opposite to the inertial rotation (in the rotating coordinate system) and substantially rely on radius-dependence of the corresponding eigenvalues. Furthermore, Bryan's factors of the acoustically loaded spheres are higher than the corresponding Bryan's factors of unloaded spheres. For example, in the case $n=2, m=2$ for eigenvalue $f = 5056 \, Hz$ of the unloaded sphere the value of Bryan's factor is $\eta = 0.137$; the corresponding eigenvalue of the acoustically loaded sphere is $f^{(a)} = 5050 \, Hz$ with Bryan's factor $\eta^{(a)} = 0.403$. Eigenfunctions, corresponding to these eigenvalues are shown in Figure – 2.

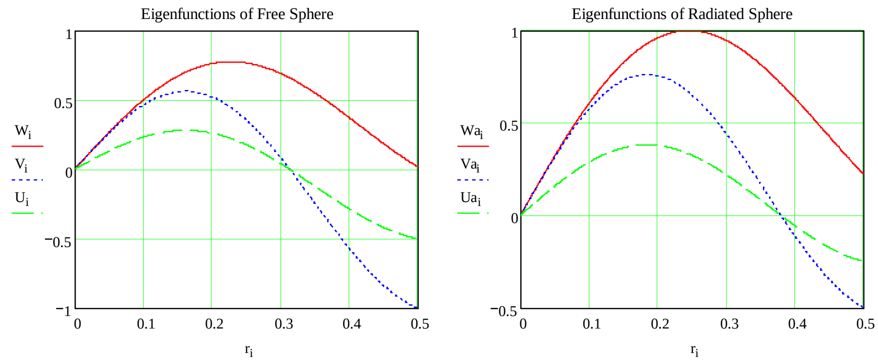

Figure 2 – Eigenvalues, corresponding to m=2, n=2, f=5056 Hz.

Bryan's factors of the torsional modes do not rely on radius dependence of their eigenfunctions and depend exclusively on values of $n,m-$ wavenumbers. In the rotating coordinate system they move in the direction the inertial rotation and do not interact with an external acoustical medium.

CONCLUSIONS

1. Expression for Bryan's factor was derived, which characterizes the coefficients of proportionality between angular rate of precession of a vibrating pattern and the inertial angular rate of the spherical isotropic elastic body;
2. It was pointed out that the Bryan's factor is an invariant of sphere's radius, its mass density and modulus of elasticity; it depends on Poisson's ratio.
3. It was found that in the case of spheroidal oscillations the Bryan's factor of radiated body is higher than the value of these factor for free body of the same mode; torsional oscillations do not interact with an ideal non-viscous acoustic medium.
4. In the rotating coordinate system the spheroidal vibrating patterns precess in the direction, which is opposite to the direction of inertial rotation (positive Bryan's factor); the torsional patterns precess in the direction of inertial rotation (negative Bryan's factor).

REFERENCES

1. A. Eringen, E.Suhubi, *Elastodynamics, Vol.2.* (Academic Press, New York, 1975).
2. G.Bryan, "On the beats in the vibrations of a revolving cylinder or bell", Proc. Cambridge Philos. Soc. Math. Phys. Sci., vol 7, 101-114 (1890).
3. V. Zhuravlev, D.Klimov, *Wave Solid-state Gyroscope.* (Nauka, Moscow, 1985, *in Russian*).