A Condition for Unbiased Direction-of-Arrival Estimation with Toeplitz Decorrelation Techniques

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***Abstract*—Toeplitz decorrelation techniques, as applied towards direction-of-arrival estimation of coherent narrowband signals using a sensor array, significantly improve the accuracy of subspace methods without reducing the effective array aperture. The novel contribution of this paper is a necessary and sufficient condition that any jointly Toeplitz and Hermitian matrix produced by these techniques must satisfy in order to facilitate unbiased estimation via a subspace method. The condition is derived for a uniform linear array and additive white Gaussian noise model. The row-selection and diagonal-averaging strategies for generating Toeplitz matrices are evaluated against the condition, for the case where all sources are coherent. It is proved that row selection accommodates unbiased estimation if and only if the first row of the deficient covariance matrix is selected, whereas diagonal averaging invariably leads to bias.**

***Index Terms*—Multiple signal classification (MUSIC), direction of arrival estimation, angle of arrival estimation, spectral estimation, Toeplitz decorrelation, coherent sources, correlated sources.**

I. Introduction

The problem of estimating the directions of arrival (DOAs) of multiple narrowband source signals incident on an array of sensors is widely encountered in practice. Different approaches towards solving this problem have appeared in the literature [1]–[3]. Amongst the proposed solutions, the subspace methods, which include the multiple signal classification (MUSIC) algorithm [4] and estimation of signal parameters via rotational invariance techniques (ESPRIT) [5], have emerged as a popular choice due to their super-resolution capability and simplicity. However, a drawback of these methods is severe performance degradation in the case of source coherence, which serves as a model for environments with multi-path propagation [6], [7]. The degradation occurs due to a reduction in the dimensionality of the eigenvector subspace, of the array covariance matrix, which corresponds to the source signals; we refer to these matrices as *deficient*.

Several approaches towards mitigating source coherence in the DOA estimation problem were proposed [7]–[10]. The most popular of these are arguably the spatial smoothing [7] and Toeplitz decorrelation techniques [8]. Whereas spatial smoothing is able to effectively 'decorrelate' source signals, this approach leads to a smaller effective array aperture, which reduces the number of sources that may be successfully resolved. In contrast, the Toeplitz techniques incur no penalty with respect to the effective array aperture, and are computationally less expensive than spatial smoothing.

Toeplitz decorrelation involves the synthesis of a full-rank covariance matrix with a Toeplitz structure from the entries of the original, deficient array covariance matrix [8]. In the ideal case, the Toeplitz matrix is constructed in a manner which guarantees consistency between the eigenvector subspace of the matrix (i.e. the subspace corresponding to the source signals) and the true DOAs. The original array covariance matrix is then replaced by the new Toeplitz matrix prior to DOA estimation. Simulations have confirmed that this approach may lead to a significant improvement in estimation accuracy [11]. However, it was found that the accuracy of the estimator is strongly dependent on the particular structure of the Toeplitz matrix; numerical results suggest that certain Toeplitz matrix structures even lead to biased estimation [12].

Toeplitz decorrelation techniques differ with respect to the structure of the Toeplitz matrix that is generated. In general, these techniques first construct a covariance vector $\mathbf{c}^T = [c_0, c_1, \ldots, c_{M-1}]$ from the elements of the deficient array covariance matrix, where M is the number of sensors in the array. A Toeplitz matrix $\mathbf{R} = [R_{i,j}]_{i,j=1}^{M}$ is subsequently generated, such that $R_{i,j} \triangleq c_{j-i}$ for all $i, j = 1, 2, \ldots, M$, and where $c_{-m} \triangleq c_m^*$. The construction of the vector $\mathbf{c}^T$ often follows one of two strategies. These strategies involve (i) assigning one of the rows of the deficient covariance matrix to $\mathbf{c}^T$ [8], [11], and (ii) averaging along the diagonals of the deficient covariance matrix, and selecting these averages as the elements of $\mathbf{c}^T$ [12]. We refer to these strategies as the *row-selection* and *diagonal-averaging* strategies, respectively.

Toeplitz techniques that correspond to each of the strategies were proposed in the literature, and results from numerical simulations were presented to substantiate the proposed structure of the corresponding Toeplitz matrices [8], [11], [12]. However, little analytical work has been carried out to evaluate whether each of the strategies, in principle, can accommodate unbiased and robust DOA estimation. In this paper, we take an initial step towards bridging this gap by deriving a necessary and sufficient condition that any jointly Toeplitz and Hermitian matrix produced by a Toeplitz technique must satisfy in order to facilitate unbiased DOA estimation using a subspace method. The derivation is carried out for a uniform linear array

(ULA) and additive white Gaussian noise (AWGN) model.

Each of the strategies towards Toeplitz matrix synthesis is evaluated against the derived condition, under the assumption that all source signals are coherent. In this scenario, it is proved that the row-selection strategy accommodates unbiased estimation if and only if the first row of the deficient covariance matrix is selected. Furthermore, it is proved that the diagonal-averaging strategy invariably leads to biased DOA estimation.

The remainder of this paper is organized as follows. In section II, we present the ULA and signal models used throughout the paper. An overview of literature pertaining to Toeplitz decorrelation techniques is presented in section III. The necessary and sufficient condition for unbiased DOA estimation is derived in section IV, and the two matrix synthesis strategies are compared against this condition in section V. The paper is concluded in section VI.

II. Uniform Linear Array and Signal Models

Consider an array of M identical omnidirectional sensors with unity gain, separated by the same distance d along a straight line. Suppose that the wavefront impinging on this ULA consists of $N < M$ narrowband source signals $s_n(t)$, each from a distinct DOA θ_n. The array output signal $\mathbf{x}(t) \triangleq [x_1(t), \ldots, x_M(t)]^T$ at time $t \in \mathbb{Z}$ is modelled as [13]

$$\mathbf{x}(t) = \sum_{n=1}^{N} \mathbf{a}(\gamma_n) s_n(t) + \mathbf{z}(t) = \mathbf{A}(\boldsymbol{\gamma})\mathbf{s}(t) + \mathbf{z}(t), \qquad (1)$$

where $\mathbf{A}(\boldsymbol{\gamma}) \triangleq [\mathbf{a}(\gamma_1), \mathbf{a}(\gamma_2), \ldots, \mathbf{a}(\gamma_N)]$ denotes the steering matrix, and $\mathbf{a}(\gamma_n) \triangleq [1, e^{-j\gamma_n}, \ldots, e^{-j(M-1)\gamma_n}]^T$ denotes the steering vector. The electrical angles $\boldsymbol{\gamma} \triangleq \{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ are given by $\gamma_n = 2\pi d \sin(\theta_n)/\lambda$, where λ is the wavelength.

The vectors $\mathbf{s}(t) \triangleq [s_1(t), s_2(t) \ldots, s_N(t)]^T$ and $\mathbf{z}(t) \triangleq [z_1(t), z_2(t) \ldots, z_M(t)]^T$ in (1) represent the source and noise signals. This research considers coherent source signals [7], which are assumed to be perfectly correlated. Let $(\Omega, \mathcal{B}, P)$ denote a probability space, where Ω denotes the sample space, $\mathcal{B}$ is a σ-algebra on Ω, and P is a probability measure. Let $\tilde{s}(t) \equiv \tilde{s}(\omega, t)$ denote a zero-mean, wide-sense stationary and narrowband random signal on this probability space, where $\omega \in \Omega$, such that $E[\tilde{s}(t)\tilde{s}^*(t)] = 1$. The source signals are modelled as

$$s_n(t) = b_n e^{j\vartheta_n} \tilde{s}(t), \qquad (2)$$

where $b_n > 0$ and $\vartheta_n \in [-\pi, \pi)$ for all n. The noise signals are modelled as zero-mean and mutually uncorrelated AWGN processes, each with a variance of σ_N^2. It is assumed that the noise signals are not correlated with the signals $s_n(t)$.

The $M \times M$ array covariance matrix $\mathbf{R}_{\mathrm{XX}} \triangleq E[\mathbf{x}(t)\mathbf{x}^H(t)]$ is given by

$$\mathbf{R}_{\mathrm{XX}} = \mathbf{A}(\gamma)\mathbf{R}_{\mathrm{SS}}\mathbf{A}^H(\gamma) + \sigma_N^2 \boldsymbol{I}, \qquad (3)$$

where $\mathbf{R}_{\mathrm{SS}} \triangleq E[\mathbf{s}(t)\mathbf{s}^H(t)]$ denotes the $N \times N$ source covariance matrix, with elements $r_{m,n} = b_m b_n \exp(j\Delta_{m,n})$, and where $\Delta_{m,n} \triangleq \vartheta_m - \vartheta_n$. In this scenario, the dimensionality of the eigenvector subspace of $\mathbf{R}_{\mathrm{XX}}$ corresponding to the source signals is reduced to one. We follow the common practice of assuming[1] that $\Delta_{m,n} = 0$ for all m and n, which implies that $r_{m,n} = b_m b_n$ (see, for example, [11], [14]).

III. Literature overview

Recent literature on Toeplitz decorrelation techniques focus on generalizing the fundamental approach to two-dimensional arrays [15] and to multiple-input multiple-output systems [16]. We consider the one-dimensional case in this paper as a starting point towards a more general analysis.

Kung et al. [10] proposed a robust method for estimating the DOAs of coherent sources using a long uniform sensor array. The authors interpret the array measurements as the output of a self-generating autoregressive moving-average process. A state space representation of the process is computed from a reduced-order Toeplitz approximation of a spatial covariance matrix estimate, which is obtained via joint spatial and temporal averaging. The DOAs are subsequently estimated by finding the spatial frequencies corresponding to the maxima of the spatial power spectrum. Numerical simulations presented by the authors suggest that the proposed method produces accurate DOA estimates when $N \ll M$; however, the case where N approaches M was not investigated.

Han and Zhang [8] proposed a row–selection technique for constructing a Toeplitz matrix replacement for the deficient array covariance matrix. The technique assigns any row[2] of the deficient array covariance matrix to the covariance vector $\mathbf{c}^T$. DOA estimation is performed using ESPRIT [5]; it was shown both analytically and numerically that the proposed technique leads to unbiased DOA estimation using a ULA, irrespective of the coherency of the source signals.

Hui et al. [11] proposed the cross-correlation vector Toeplitz reconstruction (CVT) technique. This technique assigns the first row of the deficient array covariance matrix to the covariance vector $\mathbf{c}^T$. In the case of coherent sources, it was shown analytically that CVT achieves an improvement in the effective signal-to-noise power ratio pertaining to the source and noise eigenvector subspaces of the reconstructed matrix. Numerical experiments involving DOA estimation of coherent sources using a ULA were performed; it was shown that DOA estimation using the reconstructed Toeplitz matrix is superior, in terms of the probability of successful estimation, root-mean-square error and effective array aperture, as compared to DOA estimation with spatial smoothing.

Jun et al. [12] proposed a Toeplitz decorrelation technique in which the elements of the covariance vector $\mathbf{c}^T$ correspond to averages computed over the diagonals of the deficient array covariance matrix. Three Toeplitz matrices were proposed, corresponding to (i) direct averaging of the entries in each diagonal, (ii) averaging of the magnitude of the entries, while retaining the arguments of the individual entries, and (iii) averaging of the arguments of the entries, while retaining the magnitudes of the individual entries. Numerical results

[1]The more general case is to be considered in a future publication.

[2]It is worth noting that, if any row other than the first is used in this context, the Toeplitz matrix is no longer Hermitian; yet, the source and noise signal subspaces are preserved.

presented by the authors suggest that the direct averaging approach leads to biased DOA estimation.

IV. A condition for unbiased estimation

Let $\tilde{\mathbf{R}}_{\mathrm{XX}}$ denote an $M \times M$ Toeplitz matrix obtained using a Toeplitz decorrelation technique. We consider the case where $\tilde{\mathbf{R}}_{\mathrm{XX}}$ is of full rank and Hermitian. It follows that $\tilde{\mathbf{R}}_{\mathrm{XX}}$ has a spectral decomposition; let the eigenvalues be denoted by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M$, and let the corresponding orthogonal eigenvectors be denoted by $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_M$.

We consider the case where DOA estimation is performed by applying a subspace method to the Toeplitz matrix $\tilde{\mathbf{R}}_{\mathrm{XX}}$, which replaces the deficient array covariance matrix $\mathbf{R}_{\mathrm{XX}}$. In practice, the number of source signals N is unknown a priori, and has to be estimated from measurements. A common method for estimating N selects the dimensionality of the noise subspace as the estimated number $\hat{K}$ of eigenvalues of $\tilde{\mathbf{R}}_{\mathrm{XX}}$ that are of smallest and equal[3] magnitude λ_M [17]. The estimated number $\hat{N}$ of source signals is then selected as $M - \hat{K}$. This method is incorporated into the DOA estimation scenario considered here. Since the present analysis considers a symbolic expression for the Toeplitz matrix $\tilde{\mathbf{R}}_{\mathrm{XX}}$, the true number K of eigenvalues that are equal to λ_M is available. Therefore, during DOA estimation, the signal subspace $\mathcal{S}$ is selected as $\mathcal{S} = \mathrm{sp}\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_{M-K}\}$, whereas the noise subspace $\mathcal{N}$ is selected as $\mathcal{N} = \mathrm{sp}\{\boldsymbol{\mu}_{M-K+1}, \ldots, \boldsymbol{\mu}_M\}$ (here, $\mathrm{sp}\{\cdot\}$ denotes the span of the vector arguments). Accordingly, we declare the matrix $\tilde{\mathbf{R}}_{\mathrm{XX}}$ as accommodating *unbiased* DOA estimation using any subspace method if it satisfies

$$\mathrm{sp}\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_{M-K}\} = \mathrm{sp}\{\mathbf{a}(\gamma_1), \ldots, \mathbf{a}(\gamma_N)\} \tag{4}$$

and

$$\mathbf{a}(\gamma_n)^H \boldsymbol{\mu}_m = 0, \tag{5}$$

for all $M - K < m \leq M$ and $n = 1, 2, \ldots, N$.

A necessary and sufficient condition for $\tilde{\mathbf{R}}_{\mathrm{XX}}$ to accommodate unbiased DOA estimation with any subspace method is subsequently derived. Consider the matrix $\mathbf{Q} \triangleq \tilde{\mathbf{R}}_{\mathrm{XX}} - \lambda_M \mathbf{I}$. This Hermitian matrix has eigenvalues $\lambda_m - \lambda_M$, where $m = 1, 2, \ldots, M$, and the same eigenvectors as $\tilde{\mathbf{R}}_{\mathrm{XX}}$. It follows that $\mathbf{Q}$ has K eigenvalues equal to zero, and therefore a rank of $M - K$. The Vandermonde decomposition [18] of this matrix is given by $\mathbf{Q} = \mathbf{A}(\boldsymbol{\phi})\mathbf{D}\mathbf{A}^H(\boldsymbol{\phi})$, where $\mathbf{A}(\boldsymbol{\phi})$ denotes the $M \times (M - K)$ steering matrix with respect to a set of distinct angles $\boldsymbol{\phi} = \{\phi_1, \phi_2, \ldots, \phi_{M-K}\}$, and $\mathbf{D}$ is a real diagonal matrix with strictly positive diagonal entries. It follows that $\tilde{\mathbf{R}}_{\mathrm{XX}}$ may always be decomposed as

$$\tilde{\mathbf{R}}_{\mathrm{XX}} = \mathbf{A}(\boldsymbol{\phi})\mathbf{D}\mathbf{A}^H(\boldsymbol{\phi}) + \lambda_M \mathbf{I}. \tag{6}$$

The decomposition of (6) is unique with respect to the values of K and λ_M, as well as the angles $\boldsymbol{\phi}$ and the entries of $\mathbf{D}$.

It is subsequently proved that $M - K = N$ and $\boldsymbol{\phi} = \boldsymbol{\gamma}$ if $\tilde{\mathbf{R}}_{\mathrm{XX}}$ accommodates unbiased estimation. Since the column rank of $\mathbf{A}(\boldsymbol{\gamma})$ is equal to N, (4) implies that $M - K \geq N$. However, the vectors $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_{M-K}$ are orthogonal. Hence, the column rank of $[\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_{M-K}]$ is equal to $M - K$. It follows directly from (4) that $M - K = N$.

[3] In practice, only an estimate of the Toeplitz matrix $\tilde{\mathbf{R}}_{\mathrm{XX}}$ is available; hence, the computed eigenvalues may at best be approximately equal.

Now consider the eigenvectors with corresponding eigenvalues equal to λ_M. These eigenvectors satisfy

$$\mathbf{Q}\boldsymbol{\mu}_m = \mathbf{A}(\phi)\mathbf{D}\mathbf{A}(\phi)^H \boldsymbol{\mu}_m = 0 \tag{7}$$

for all $m = N+1, N+2, \ldots, M$. Since $\mathbf{A}(\phi)$ is of full column rank and $\mathbf{D}$ is of full rank, it follows that $\mathbf{A}(\phi)^H \boldsymbol{\mu}_m = \mathbf{0}$ and

$$\mathbf{a}(\phi_n)^H \boldsymbol{\mu}_m = 0 \tag{8}$$

for all $n = 1, 2, \ldots, N$. These eigenvectors, which span the noise subspace, are orthogonal to the signal subspace spanned by the eigenvectors $\boldsymbol{\mu}_i$, where $i = 1, 2, \ldots, N$. Hence, from (8), we have

$$\mathrm{sp}\{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_N\} = \mathrm{sp}\{\mathbf{a}(\phi_1), \ldots, \mathbf{a}(\phi_N)\}. \tag{9}$$

It subsequently follows from (4) that $\mathrm{sp}\{\mathbf{a}(\phi_1), \ldots, \mathbf{a}(\phi_N)\} = \mathrm{sp}\{\mathbf{a}(\gamma_1), \ldots, \mathbf{a}(\gamma_N)\}$. This implies that the vector $\mathbf{a}(\phi_m) \in \mathrm{sp}\{\mathbf{a}(\gamma_1), \ldots, \mathbf{a}(\gamma_N)\}$ for every $m = 1, 2, \ldots, N$. However, $\mathbf{a}(\phi_m) \notin \mathrm{sp}\{\mathbf{a}(\gamma_1), \ldots, \mathbf{a}(\gamma_N)\}$ if $\phi_m \neq \gamma_j$ for any $j = 1, 2, \ldots, N$. It then follows that $\phi_m = \gamma_j$ must hold for some value of j. Repeating this for all angles in $\boldsymbol{\phi}$ leads to the conclusion that $\boldsymbol{\phi} = \boldsymbol{\gamma}$. We have therefore proved that any Hermitian and Toeplitz matrix $\tilde{\mathbf{R}}_{\mathrm{XX}}$ that accommodates unbiased estimation must have a decomposition

$$\tilde{\mathbf{R}}_{\mathrm{XX}} = \mathbf{A}(\boldsymbol{\gamma})\mathbf{D}\mathbf{A}^H(\boldsymbol{\gamma}) + \lambda_M \mathbf{I}. \tag{10}$$

We now prove that $\tilde{\mathbf{R}}_{\mathrm{XX}}$, as defined at the start of this section, necessarily accommodates unbiased DOA estimation using a subspace technique if it has the particular decomposition of (10). Since $\tilde{\mathbf{R}}_{\mathrm{XX}}$ has K eigenvalues of minimum and equal magnitude, (10) implies that $M - K = N$. Then using the same arguments as were used to derive (7) to (9), but where $\boldsymbol{\phi}$ is substituted with $\boldsymbol{\gamma}$, both (4) and (5) are shown to hold. $\tilde{\mathbf{R}}_{\mathrm{XX}}$ therefore accommodates unbiased DOA estimation.

We have proved that any full-rank jointly Toeplitz and Hermitian matrix $\tilde{\mathbf{R}}_{\mathrm{XX}}$ accommodates unbiased DOA estimation, as defined at the start of this section, if and only if $\tilde{\mathbf{R}}_{\mathrm{XX}}$ has the decomposition of (10). Since the first row of a jointly Toeplitz and Hermitian matrix determines the remaining entries of the matrix, an equivalent condition may be derived from (10). The equivalent condition requires that the first row $\tilde{\mathbf{r}}^T$ of $\tilde{\mathbf{R}}_{\mathrm{XX}}$ must be expressible as

$$\tilde{\mathbf{r}}^T = \sum_{n=1}^{N} d_n \mathbf{a}(\gamma_n)^H + \lambda_M \mathbf{u}_1^T, \tag{11}$$

where $\mathbf{D} \triangleq \mathrm{diag}(d_1, d_2, \ldots, d_N)$, such that each $d_n > 0$, and $\mathbf{u}_1 \triangleq [1, 0, 0, \ldots, 0]^T$, with $\lambda_M > 0$.

V. Evaluation of matrix synthesis strategies

Two strategies for Toeplitz matrix synthesis from the literature are evaluated to determine whether they accommodate unbiased DOA estimation using subspace methods. These are the row-selection [8], [11] and diagonal-averaging [12] strategies, as described in section I. The case where all source signals are coherent is considered. Each strategy is evaluated by deriving a general expression for the first row of the Toeplitz matrix obtained using the strategy, and comparing the expression against (11).

A. Row-selection strategy

Consider any Toeplitz decorrelation technique that assigns the k^{th} row of the deficient array covariance matrix to the first row $\tilde{\mathbf{r}}^T$ of the Toeplitz matrix $\tilde{\mathbf{R}}_{\text{XX}}$. It follows that

$$\tilde{\mathbf{r}}^T = \sum_{n=1}^{N} \left[\sum_{m=1}^{N} b_m e^{-j(k-1)\gamma_m} \right] b_n \mathbf{a}(\gamma_n)^H + \sigma_N^2 \mathbf{u}_k^T, \quad (12)$$

where $\mathbf{u}_k^T$ is a vector containing zeros, but with the exception of the k^{th} element, which is equal to unity. Now consider the case where $k = 1$ – i.e., where the first row of the deficient matrix is used. It follows that the bracketed term in (12) reduces to $c = \sum_{m=1}^{N} b_m$, which is a positive real number. Hence, (12) matches (11) with $d_n = cb_n$ and $\lambda_M = \sigma_N^2$. This result confirms that the strategy of selecting the first row accommodates unbiased DOA estimation. However, in the case where $k > 1$, the bracketed term of (12), which is a sum of complex numbers with arbitrary arguments, cannot generally be assumed as being equal to a positive real number. Hence, (12) is inconsistent with (11). It is concluded that the use of any row other than the first leads to biased DOA estimation[4].

B. Diagonal-averaging strategy

Consider the strategy which involves direct averaging of the diagonals in the upper triangular region of the deficient array covariance matrix, and assigning these values to the first row $\tilde{\mathbf{r}}^T$ of the Toeplitz matrix $\tilde{\mathbf{R}}_{\text{XX}}$. The average of the k^{th} diagonal of the deficient matrix, where $k = 0$ denotes the main diagonal, is given by

$$\mu_k = \sum_{n=1}^{N} \left[\sum_{m=1}^{N} C(m, n) \right] b_n e^{jk\gamma_n} + \sigma_N^2 \delta_{k,0}, \quad (13)$$

where $k = 0, 1, \ldots, M - 1$ and $\delta_{k,0}$ denotes the Kronecker delta function, and where

$$C(m, n) = \frac{b_m}{M - k} \left[\sum_{p=1}^{M-k} e^{-j(p-1)(\gamma_m - \gamma_n)} \right]. \quad (14)$$

Let $\tilde{\mathbf{r}}^T = [\mu_0, \mu_1, \ldots, \mu_{M-1}]$. It follows that

$$\tilde{\mathbf{r}}^T = \sum_{n=1}^{N} D(n) b_n \mathbf{a}(\gamma_n)^H + \sigma_N^2 \mathbf{u}_1^T, \quad (15)$$

where $D(n) = \sum_{m=1}^{N} C(m, n)$. Since it cannot generally be assumed that $D(n)$ is a real number, a comparison of (15) with (11) leads to the conclusion that the diagonal-averaging strategy considered here invariably leads to biased DOA estimation.

[4]The noise term is also inconsistent – i.e., $\mathbf{u}_1 \neq \mathbf{u}_k$ if $k > 2$.

VI. CONCLUSION

In this paper, we presented a necessary and sufficient condition for a Toeplitz decorrelation technique to accommodate unbiased DOA estimation using a subspace method. Two common strategies towards constructing the Toeplitz matrix were evaluated against this condition. It is concluded that the row-selection strategy accommodates unbiased estimation if and only if the first row is selected, whereas the diagonal-averaging approach invariably leads to biased estimation.

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