Construction of Semi-Markov Ergodic Maps with Selectable Spectral Characteristics via the Solution of the Inverse Eigenvalue Problem

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Abstract—**This paper presents a novel technique for constructing semi–Markov ergodic maps that harnesses a new solution of the inverse eigenvalue problem for 3–by–3 doubly stochastic matrices. The proposed solution facilitates the selection of the spectral characteristics of the trajectories generated by these ergodic maps through the selection of appropriate eigenvalues for the Frobenius–Perron matrices associated with the maps. It is proved that the proposed solution is able to realise all possible eigenvalue triples for the matrices of interest, thereby providing greater freedom in selecting the power spectral density than existing techniques. The novel technique is demonstrated by constructing several semi–Markov ergodic maps with distinct power spectra. It is concluded that the flexibility and versatility of the technique holds potential for the purpose of system modelling in various contexts.**

I. Introduction

Dynamical systems encountered in various contexts are known to exhibit chaotic behaviour [1]. It has been shown that chaotic dynamics may arise in systems such as electrical circuits, solid state devices, lasers and mechanical devices [2]–[5], in addition to systems from fields such as biology, chemistry and economics [6], [7]. Signals arising from these systems appear to fluctuate randomly, due to the systems' sensitive dependence on the initial system state; this characteristic complicates the effective processing of these signals.

Accurate modelling of chaotic systems is a prerequisite for developing optimal strategies for processing signals arising from these systems. Specifically, accurate models of chaotic systems facilitate improved prediction, statistical inference and system control [8]–[10]. As such, the problem of modelling the evolution rule of an unknown chaotic system from the observed dynamical behaviour and statistical properties of the system is of importance. The inverse Frobenius–Perron (FP) problem, which addresses the design of an ergodic dynamical system's evolution rule such that the invariant probability distribution of its state matches a prescribed distribution, may be used as a starting point for system modelling in this context.

In designing an ergodic map to model an unknown chaotic system's evolution rule, the flexibility to specify both the invariant probability distribution associated with the system state as well as the time autocorrelation function (ACF) corresponding to state trajectories, as would result from the choice of the ergodic map, is of interest. Several authors have proposed solutions to the inverse FP problem that also facilitate the prescription of the ACF [11]–[15]. However, these solutions have certain shortcomings. Solutions that use stochastic algorithms to derive the ergodic map [11] provide little insight as to how the structure of the resulting map gives rise to the ACF. Whereas analytic solutions have been proposed that shed light on this relationship [12], [15], these solutions only provide limited control over the resulting ACF.

This paper presents a generalisation of the solution to the inverse FP problem presented in [15], which provides a means for constructing a semi–Markov ergodic map such that a prescribed invariant probability distribution is approximated while at the same time providing control over the resulting ACF. It was shown that the solution of [15] can only realise ACFs that correspond to FP matrices[1] with real eigenvalues. Mori et al. [16] proved that the time ACF associated with an ergodic map may be expressed as a linear combination of component functions that correspond to the FP operator eigenvalues, where the argument of each eigenvalue is equal to the frequency of oscillation of the corresponding component function. Hence, it is anticipated that the set of ACF component functions realisable by the solution of [15] is limited to exponentially decaying functions that either decrease monotonically, or oscillate at a frequency of π radians per sample. This limitation, which was demonstrated experimentally in [15], restricts the practicality of the solution.

The current paper presents a solution to the inverse eigenvalue problem for constructing 3–by–3 doubly stochastic FP matrices with prescribed sets of complex eigenvalues, and the subsequent derivation of a semi–Markov ergodic map that possesses the required FP matrix. In the case of complex eigenvalues, it is demonstrated that the ACFs attainable using this method contain component functions that oscillate with frequencies equal to eigenvalue arguments (this is consistent with the result anticipated from the theory developed in

[1]The FP matrix is the matrix representation of the FP operator, which characterises the evolution of the state probability distribution under the evaluation of the system's evolution rule.

[16]). Furthermore, it is proved that the proposed solution to the inverse eigenvalue problem realises all possible eigenvalue triples for doubly stochastic matrices, thereby providing greater freedom in selecting the power spectrum of trajectories than the existing solution to the inverse FP problem for semi–Markov maps [15]. Whereas the proposed technique is limited to uniform invariant state distributions, it is anticipated that the 3–by–3 matrices may be used to construct higher–order maps such that a prescribed invariant distribution is approximated while reserving greater flexibility with regards to the ACF.

The remainder of this paper is set out as follows. In section II, an overview of existing solutions to the inverse FP problem that facilitate the specification of the ACF is presented. Several definitions and preliminary results related to the ergodic maps of interest are provided in section III. In section IV, the proposed solution to the inverse eigenvalue problem, as well as the proposed technique for constructing the ergodic map, is presented. The proposed technique's ability to realise systems with distinct power spectra is demonstrated through experimentation in section V. The paper is concluded in section VI.

II. Literature review

An overview of existing solutions to the inverse FP problem is provided in [15]. Rogers et al. [17] proposed a technique for constructing semi–Markov ergodic maps with prescribed invariant probability density functions (PDFs) that are piecewise constant. This technique constructs the map directly from an N–by–N FP matrix with a predefined structure, which is fully specified by a set of $2N$ parameters. The structure of the FP matrix leads to a simplified analytic expression for the Perron eigenvector (i.e. the eigenvector associated with the unity eigenvalue), which coincides with the invariant PDF of the map. It was demonstrated that an arbitrary piecewise constant invariant PDF may be selected through a suitable choice of the FP matrix parameters. The relationship between the parameters and the rate of decay of the ACF was investigated, but the analysis is limited to positive and monotonically decreasing ACFs. Furthermore, it is not indicated how individual ACF component functions may be selected using the method.

Diakonos et al. [11] proposed a stochastic algorithm for generating unimodal maps with prescribed invariant PDF and ACF. Whereas this proposed technique provides a large degree of flexibility in specifying the ACF, it is computationally intensive. Furthermore, in contrast to analytic solutions of the inverse FP problem, the technique may fail to converge or produce an accurate solution with regards to the ACF.

Baranovsky and Daems [12] considered the design of ergodic maps with prescribed invariant distributions and ACFs. The technique involves the design of an initial piecewise linear map with uniform invariant PDF and a predistorted ACF. The required map is subsequently obtained via a conjugation transformation. Whereas the technique allows for the construction of initial maps with ACFs having richer properties as compared to Markov maps, the proposed technique is limited in that it only allows for the selection of conjugate map ACFs with a restricted form (i.e. the conjugate map's normalised ACF at delays $\tau \geq 1$ is necessarily equal to the initial map's normalised ACF at delay $\tau = 1$, raised to the τth power).

Nie and Coca [13], [14] proposed a technique for constructing piecewise linear semi–Markov maps that approximate the evolution of an unknown system from a sequence of PDFs generated by the system. Whereas the proposed technique is able to capture the dynamical behaviour of the system, it requires the generation of PDFs by selecting the initial state of the system, which is not possible in certain contexts.

McDonald and Van Wyk [15] proposed a solution to the inverse FP problem for constructing a semi–Markov ergodic map such that the map's invariant distribution approximates a prescribed distribution. The solution involves the construction of a stochastic matrix with a prescribed eigenspectrum via recursive Markov state disaggregation [18], and the subsequent derivation of a semi–Markov map with FP matrix equal to the stochastic matrix. The proposed solution grants a certain degree of freedom in selecting the ACF component functions. However, the recursive Markov state disaggregation technique is limited to producing stochastic matrices with real eigenvalues; it was demonstrated experimentally that the corresponding ACF component functions are limited to exponentially decaying functions that either decreasing monotonically, or oscillate at a frequency of π radians per sample.

III. Preliminaries

The current paper uses the notation of [15]. Consider a nonlinear map $S : \mathcal{I} \to \mathcal{I}$, where $\mathcal{I} = [a, b]$ denotes a compact interval of the real line. Let S be measurable and nonsingular with respect to the Borel σ–algebra on $\mathcal{I}$ and the normalised Lebesgue measure. Furthermore, let X_0 denote a random variable (RV) on $\mathcal{I}$ with an absolutely continuous distribution and PDF f_0. The evaluation of the map S according to the expression $X_{i+1} = S(X_i)$, for $i \in \{0, 1, \ldots\}$, produces a sequence of RVs $\{X_1, X_2, \ldots\}$ with corresponding PDFs given by $f_{i+1}(x) = \mathcal{P}_S[f_i(x)]$. In this expression, $\mathcal{P}_S$ is the FP operator associated with S [19]. If the PDF f_i associated with the RV X_i asymptotically converges to a unique invariant PDF $f_S^*(x)$ such that $f_S^*(x) = \mathcal{P}_S[f_S^*(x)]$, then S is ergodic.

In the remainder of this paper, ergodic maps S with unique invariant densities and that belong to the class of semi–Markov maps are considered. Semi–Markov maps [20], which constitute a superset of the class of Markov maps, are defined in what follows. Let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_N\}$ denote a partition of $\mathcal{I} = [a, b]$ into N nonoverlapping intervals, such that $Q_n = [q_{n-1}, q_n)$ for $n = 1, 2, \ldots, N-1$, $Q_N = [q_{N-1}, b]$ and $q_0 = a$. A map S belongs to the class of $\mathcal{Q}$–semi–Markov maps if there exist disjoint intervals $R_j^{(n)}$ such that, for any $n = 1, 2, \ldots N$, $Q_n = \cup_{j=1}^{k(n)} R_j^{(n)}$, $S|_{R_j^{(n)}}$ is monotonic, and $S(R_j^{(n)}) \in \mathcal{Q}$. It was proved in [20] that the invariant PDF f_S^* of a piecewise linear and expanding $\mathcal{Q}$–semi–Markov map (i.e. a map where $S|_{R_j^{(n)}}$ is linear with a slope having an absolute value greater than unity, for all $n = 1, 2, \ldots N$ and $j = 1, 2, \ldots k(n)$) is piecewise constant on the intervals of $\mathcal{Q}$.

Consider the restriction of the FP operator of a piecewise linear and expanding $\mathcal{Q}$–semi–Markov map S to the space of functions constant on the intervals of $\mathcal{Q}$. Furthermore, let PDFs with domain $\mathcal{I}$ that belong to this space be represented by row vectors $\boldsymbol{f}$ of length N, such that each vector element equals the constant value of the PDF over the corresponding interval of $\mathcal{Q}$. This restriction facilitates the representation of the FP operator $\mathcal{P}_S$ as an N–by–N matrix $\mathbf{P}_S$, which is referred to as the FP matrix of S, such that $\boldsymbol{f}_{i+1} = \boldsymbol{f}_i \mathbf{P}_S$. The invariant density $\boldsymbol{f}_S^*$ corresponding to the map S is the left eigenvector of the matrix $\mathbf{P}_S$ that corresponds to the eigenvalue of unity (this follows from the expression $\boldsymbol{f}_S^* = \boldsymbol{f}_S^* \mathbf{P}_S$). The FP matrix $\mathbf{P}_S = [P_{i,j}]_{i,j=1,2,\ldots,N}$ may be derived from its corresponding $\mathcal{Q}$–semi–Markov map by setting

$$P_{i,j} = \left\{ \begin{array}{cc} |(S|_{R_k^{(i)}})'|^{-1} & \text{if } S(R_k^{(i)}) = Q_j \\ 0 & \text{otherwise} \end{array} \right\}. \quad (1)$$

Consider any N–by–N stochastic matrix $\mathbf{P}$ (i.e. a matrix with real elements restricted to the interval $[0,1]$, and with rows that sum to unity). Gora and Boyarsky [20] proved the existence of a piecewise linear and expanding semi–Markov map defined over the N–interval uniform partition $\mathcal{U}$ of the interval $\mathcal{I}$, such that the FP matrix associated with the map is equal to the stochastic matrix $\mathbf{P}$. An algorithm for constructing a $\mathcal{U}$–semi–Markov ergodic map with this property is provided in [20] (proposition 1). This algorithm is a component of the proposed technique for constructing semi–Markov maps with selectable spectrum; it is provided in what follows.

Algorithm 1. *(Construction of a piecewise linear and expanding $\mathcal{U}$–semi–Markov ergodic map with FP matrix equal to stochastic matrix $\mathbf{P}$ [20]): Let $u_n = a + n(b-a)/N$. Furthermore, let $U_n = [u_{n-1}, u_n)$ where $n = 1, 2, \ldots, N-1$, and let $U_N = [u_{N-1}, u_N]$. For each interval U_n, the function $S|_{U_n}$ is constructed. Consider the FP matrix elements $P_{n,j_1}, P_{n,j_2}, \ldots, P_{n,j_k} > 0$, such that $P_{n,j_1} + P_{n,j_2} + \ldots + P_{n,j_k} = 1$. Compute the intervals $R_s^{(n)} = [r_{s-1}^{(n)}, r_s^{(n)})$, where $s = 1, 2, \ldots, k$ and*

$$r_s^{(n)} = u_{n-1} + \frac{(b-a)}{N} \sum_{v=1}^{s} P_{n,j_v}. \quad (2)$$

The function $S|_{R_s^{(n)}}$, for $s = 1, 2, \ldots k$, is given by

$$S|_{R_s^{(n)}}(x) = (x - r_{s-1}^{(n)})/(P_{n,j_s}) + u_{j_s}. \quad (3)$$

This section is concluded with the characterisation of the time ACF associated with the trajectories of an ergodic map, as presented by Mori et al. [16]. In general, for an ergodic map S with FP operator $\mathcal{P}_S$, the time ACF is given by

$$\begin{aligned} \phi(\tau) &= \sum_{n=1}^{\infty} b_n \lambda_n^{\tau} \\ &= \sum_{n=1}^{\infty} b_n \exp[(\ln(|\lambda_n|) + i \arg(\lambda_n))\tau], \end{aligned} \quad (4)$$

where b_n depends on the eigenfunctions of $\mathcal{P}_S$, λ_n denotes the nth eigenvalue of $\mathcal{P}_S$, and $i \triangleq \sqrt{-1}$. In general, the FP operator may have both real and complex eigenvalues, where $\lambda_1 = 1$ and $|\lambda_n| \leq 1$ for all $n > 1$. Eq. 4 reveals that the normalised ACF is a linear combination of oscillating and exponentially damped component functions. The rate of decay of each component is determined by the magnitude of the corresponding eigenvalue, whereas its oscillation frequency is determined by the argument of the corresponding eigenvalue.

IV. Methods

The solution of the inverse eigenvalue problem for 3–by–3 doubly stochastic matrices is presented in two parts, corresponding to the case of real eigenvalues and the case of strictly complex eigenvalues[2]. It is proved that this solution realises all possible eigenvalue triplets for these matrices. The section concludes with an algorithm for constructing a semi–Markov ergodic map with selectable spectral characteristics.

A. Inverse Eigenvalue Problem: Real Eigenvalues

The inverse problem for constructing a doubly stochastic matrix with prescribed real eigenvalues is solved using recursive Markov state disaggregation (MSD) [15], [18]. Recursive MSD is a technique for constructing a stochastic matrix by interpreting it as the transition matrix of a Markov chain. Starting with an elementary single–state Markov chain, the states of the Markov chain are disaggregated (or split) one–by–one in a recursive fashion. The disaggregation of a particular state is achieved by increasing the dimensionality of the transition matrix of the Markov chain and recomputing a subset of the transition probabilities in a particular manner.

Let $s_k^{(j-1)}$ denote a specific state of a Markov chain immediately prior to the jth application of MSD, and let its corresponding stationary probability be denoted by $p_k^{(j-1)}$, where $j = 1, 2, \ldots N_D$. Consider a case where state $s_k^{(j-1)}$ is to be disaggregated during the jth application of MSD. To achieve disaggregation, select elements of the transition matrix $\mathbf{P}^{(j-1)}$ associated with the preceding Markov chain are recomputed in such a manner that the stationary probabilities of those states that are not currently undergoing disaggregation remain the same. The stationary probability of the preceding state $s_k^{(j-1)}$ is divided between the two new states $s_{k_1}^{(j)}$ and $s_{k_2}^{(j)}$ such that $p_{k_1}^{(j)} = \alpha^{(j)} p_k^{(j-1)}$ and $p_{k_2}^{(j)} = (1 - \alpha^{(j)}) p_k^{(j-1)}$, where $\alpha^{(j)} \in (0, 1)$ is specified prior to the jth application of MSD. Furthermore, the matrix $\mathbf{P}^{(j)}$ of the successive Markov chain (i) inherits all eigenvalues of the corresponding transition matrix $\mathbf{P}^{(j-1)}$, and (ii) contains an additional eigenvalue $\lambda^{(j)}$ that is specified prior to the jth application of MSD.

MSD is used to solve the inverse problem for real eigenvalues as follows. All doubly stochastic matrices $\mathbf{P}$ have an eigenvalue of unity; without loss of generality, let $\lambda_1 = 1$ and $\lambda_2, \lambda_3 \in \mathbb{R}(-1, 1)$ denote the remaining eigenvalues of the

[2] All doubly stochastic matrices possess an eigenvalue λ_1 equal to unity. The two cases considered in this section (real and strictly complex eigenvalues) pertain to the remaining eigenvalues λ_2 and λ_3 of the 3–by–3 matrix.

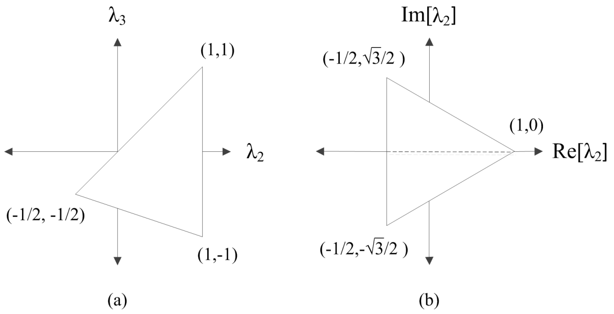

Fig. 1. Regions of specifiable eigenvalues for the proposed solution to the inverse eigenvalue problem, in the (a) real and (b) strictly complex cases.

matrix[3] that is to be derived, with the requirement that

$$\lambda_3 \leq \lambda_2 \leq 1. \tag{5}$$

MSD is first applied to the single state of the elementary Markov chain with $\alpha^{(1)} = 2/3$ and $\lambda^{(1)} = \lambda_2$, thereby obtaining a two–state Markov chain with states $s_1^{(1)}$ and $s_2^{(1)}$. The stationary probabilities of these states are equal to $p_1^{(1)} = 2/3$ and $p_2^{(1)} = 1/3$, whereas the transition matrix $\mathbf{P}^{(1)}$ eigenvalues are equal to $\lambda_1 = 1$ and λ_2. In the subsequent step, state $s_1^{(1)}$ is disaggregated with $\alpha^{(2)} = 1/2$ and $\lambda^{(2)} = \lambda_3$, thereby obtaining a Markov chain with states $s_1^{(2)}$, $s_2^{(2)}$ and $s_3^{(2)}$. The stationary probabilities of these states are all equal to 1/3, whereas the eigenvalues of the transition matrix $\mathbf{P}^{(2)}$ associated with this Markov chain are equal to $\{\lambda_1, \lambda_2, \lambda_3\}$. The transition matrix $\mathbf{P} \triangleq \mathbf{P}^{(2)}$ is given by

$$\mathbf{P} = \frac{1}{6}\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_3 & \alpha_4 \end{pmatrix}, \tag{6}$$

where

$$\alpha_1 = 2 + \lambda_2 + 3\lambda_3, \tag{7}$$
$$\alpha_2 = 2 + \lambda_2 - 3\lambda_3, \tag{8}$$
$$\alpha_3 = 2(1 - \lambda_2), \tag{9}$$
$$\alpha_4 = 2(1 + 2\lambda_2). \tag{10}$$

The requirement that the elements of the matrix must lie within the interval $[0, 1]$, combined with eq. 5, imposes the conditions

$$-1/2 \leq \lambda_2 \leq 1 \tag{11}$$

and

$$-\frac{2 + \lambda_2}{3} \leq \lambda_3 \leq \lambda_2 \tag{12}$$

on the eigenvalues that may be specified. The region of eigenvalues that may be specified using the matrix of eq. 6, as illustrated in fig. 1(a), coincides with the entire region of possible real eigenvalues for doubly stochastic matrices [21].

[3]The derivation is conducted under the assumption that there exists a doubly stochastic matrix with eigenvalues λ_2 and λ_3. At the end of the derivation, conditions on the eigenvalues are derived that guarantee the existence of the matrix.

It follows that eqs. 6 to 10, under the conditions of eqs. 11 and 12, may be used to solve the inverse eigenvalue problem for doubly stochastic matrices in the case of real eigenvalues λ_2 and λ_3.

B. Inverse Eigenvalue Problem: Strictly Complex Eigenvalues

A generalisation of the approach that was used to derive an expression for the generalised rotation matrix in [22] is used to solve the inverse eigenvalue problem for 3–by–3 doubly stochastic matrices with strictly complex eigenvalues. As in the real–valued case, let the unity eigenvalue of the doubly stochastic matrix be denoted by λ_1. Perfect and Mirsky [21] proved that the remaining eigenvalues λ_2 and λ_3, if strictly complex, must necessarily be complex conjugate pairs. Without loss of generality, let $\lambda_2 = re^{i\theta}$ and $\lambda_3 = \lambda_2^*$, where $0 < r \leq 1$ and $\theta \in (0, \pi)$. Due to the fact that the eigenvalues are distinct, the matrix $\mathbf{P}$ is diagonalisable and may be decomposed as

$$\mathbf{P} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}, \tag{13}$$

where the columns of $\mathbf{V}$ are the eigenvectors of $\mathbf{P}$, and $\mathbf{D}$ is the diagonal matrix with entries equal to the respective eigenvalues of $\mathbf{P}$. Consider the case where $\mathbf{P}$ is orthogonal. It follows that $\mathbf{V}$ is a unitary matrix,

$$\mathbf{V} = \begin{pmatrix} e^{i\phi}/\sqrt{3} & v_{1,2} & v_{1,3} \\ e^{i\phi}/\sqrt{3} & v_{2,2} & v_{2,3} \\ e^{i\phi}/\sqrt{3} & v_{3,2} & v_{3,3} \end{pmatrix}, \tag{14}$$

where $v_{i,j} \in \mathbb{C}$ and $\phi \in [0, 2\pi)$. By substitution of eq. 14 into eq. 13, and from the properties that $\mathbf{V}$ is unitary and that the elements of $\mathbf{P}$ are real–valued, the matrix $\mathbf{P}$ is derived as

$$\mathbf{P} = \frac{1}{3}\begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_3 & \beta_1 & \beta_2 \\ \beta_2 & \beta_3 & \beta_1 \end{pmatrix}, \tag{15}$$

where

$$\beta_1 = 1 + 2r\cos(\theta), \tag{16}$$
$$\beta_2 = 1 - r\cos(\theta) \mp \sqrt{3}r\sin(\theta), \tag{17}$$

and

$$\beta_3 = 1 - r\cos(\theta) \pm \sqrt{3}r\sin(\theta). \tag{18}$$

The requirement that the elements of the matrix $\mathbf{P}$ must lie within the interval $[0, 1]$ imposes the conditions

$$\mathrm{Re}\{\lambda_2\} \geq -1/2 \tag{19}$$

and

$$-\frac{1 - \mathrm{Re}\{\lambda_2\}}{\sqrt{3}} \leq \mathrm{Im}\{\lambda_2\} \leq \frac{1 - \mathrm{Re}\{\lambda_2\}}{\sqrt{3}}, \tag{20}$$

where $\mathrm{Re}\{\lambda_3\} = \mathrm{Re}\{\lambda_2\}$ and $\mathrm{Im}\{\lambda_3\} = -\mathrm{Im}\{\lambda_2\}$. The region of eigenvalues specifiable using the matrix of eq. 15, as illustrated in fig. 1(b), coincides with the entire region of possible strictly complex eigenvalues for doubly stochastic matrices [21]. It follows that eqs. 15 to 18, under the conditions of eqs. 19 and 20, may be used to solve the inverse eigenvalue problem for doubly stochastic matrices in the case of strictly complex eigenvalues λ_2 and λ_3.

TABLE I
EIGENVALUES OF FP MATRICES BELONGING TO MAPS IN $\mathcal{W}_1$ TO $\mathcal{W}_3$.

Set	Eigenvalue λ_1	Eigenvalue λ_2	Eigenvalue λ_3
$\mathcal{W}_1$	$\lambda_1 = 1$	$\lvert\lambda_2\rvert = 0.85[r_{\max}(\theta)]$ [a] $\arg[\lambda_2] \in [0, 2\pi/3]$	$\lambda_3 = \lambda_2^*$
$\mathcal{W}_2$	Map 1: $\lambda_1 = 1$ Map 2: $\lambda_1 = 1$ Map 3: $\lambda_1 = 1$ Map 4: $\lambda_1 = 1$	$\lambda_2 = 0.85$ $\lambda_2 = 0.75$ $\lambda_2 = 0.50$ $\lambda_2 = 0.05$	$\lambda_3 = 0.68$ $\lambda_3 = 0.60$ $\lambda_3 = 0.40$ $\lambda_3 = 0.04$
$\mathcal{W}_3$	$\lambda_1 = 1$	$\lvert\lambda_2\rvert \in (0, 1)$ $\arg[\lambda_2] = 2\pi/3$	$\lambda_3 = \lambda_2^*$

[a] The function $r_{\max}(\theta)$ denotes the maximum magnitude that a complex eigenvalue with argument θ may assume — refer to fig. 1(b).

C. Construction of Semi–Markov Maps with Selectable Spectral Characteristics

In what follows, a technique for deriving a $\mathcal{U}$–semi–Markov[4] ergodic map that facilitates the selection of the trajectories' spectral characteristics is presented. The proposed technique is based on the observation that the map's FP matrix eigenvalues partly determine the properties of the power spectral density (PSD) associated with the trajectories.

The first step of the technique consists of selecting suitable eigenvalues λ_2 and λ_3 for the 3–by–3 FP matrix $\mathbf{P}_S$ of the required map S, thereby selecting the spectral characteristics associated with the map. From eq. 4, it is anticipated that (i) the frequency of oscillation of each ACF component is equal to the argument of the corresponding eigenvalue of the FP matrix, and (ii) the rate of decay of the ACF component (and its bandwidth) is inversely proportional to the magnitude of the eigenvalue. These relationships are verified in section V by constructing several ergodic maps with distinct FP matrix eigenvalues, and computing the PSDs associated with the maps numerically from an ensemble of trajectories.

Having selected suitable matrix eigenvalues, the doubly stochastic FP matrix $\mathbf{P}$ is constructed using either eqs. 6 to 10 for real eigenvalues λ_2 and λ_3, or eqs. 15 to 18 for strictly complex eigenvalues λ_2 and λ_3. The final step consists of constructing the required semi–Markov map S such that the map's FP matrix $\mathbf{P}_S$ equals the matrix $\mathbf{P}$ derived in the previous step; this is achieved using algorithm 1, in which the map is constructed over a domain $\mathcal{I} = [a, b]$ that is uniformly partitioned into three intervals of equal length. It is noted that the semi–Markov map derived in the final step is not a unique solution for a prescribed FP matrix. This observation follows from the fact that each element of the FP matrix is inversely proportional to the absolute value of the slope of the linear function defined over the corresponding subinterval of the map (refer to eq. 1); hence, any of the linear functions that constitute the resulting map may be redefined with a negative slope.

V. RESULTS

The proposed technique was used to construct several ergodic semi–Markov maps in order to (i) characterise the relationship between the FP matrix eigenvalues and the spectral properties associated with the map, and (ii) illustrate how FP matrix eigenvalues may be selected to realise power spectra with distinct characteristics. The relationship between each eigenvalue's argument and the frequency of oscillation of the corresponding ACF component, as well as the relationship between each eigenvalue's magnitude and the bandwidth of the ACF component, were investigated.

[4]Recall that $\mathcal{U}$ denotes the uniform partition of the interval $[a, b]$.

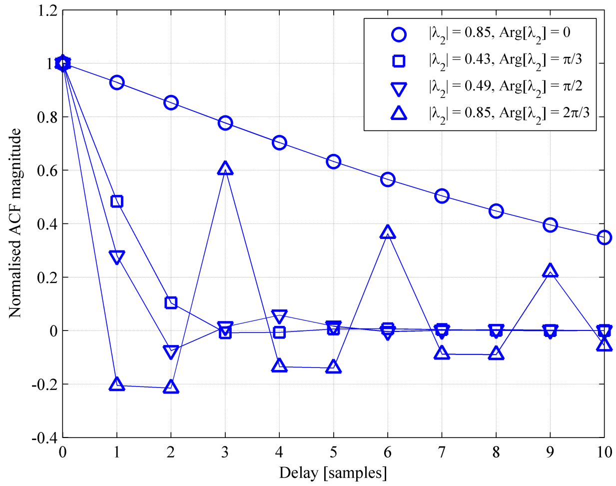

Fig. 2. Measured ACFs generated using the ergodic maps of set $\mathcal{W}_1$.

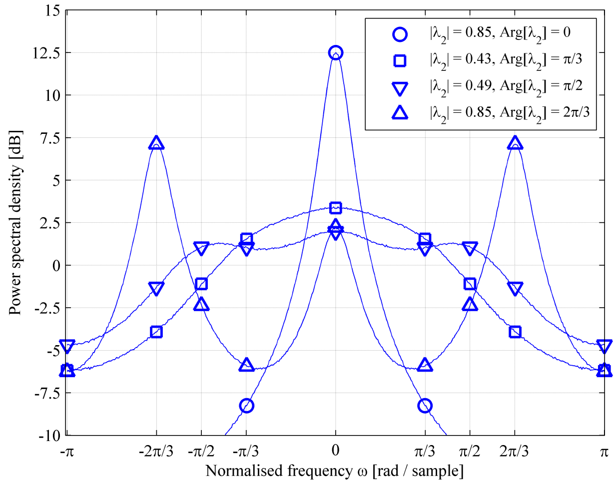

Fig. 3. Measured PSDs generated using the ergodic maps of set $\mathcal{W}_1$.

A. Eigenvalue Argument and Oscillation Frequency

A set $\mathcal{W}_1$ of $\mathcal{U}$–ergodic semi–Markov maps were derived to characterise the relationship between the FP matrix eigenvalues and the oscillation frequency of the ACF components. The eigenvalues λ_2 and $\lambda_3 = \lambda_2^*$ of the FP matrices associated with the maps in this set were selected such that $\arg[\lambda_2] \in [0, 2\pi/3]$ and $|\lambda_2| = 0.85[r_{\max}(\theta)]$, where $r_{\max}(\theta)$ denotes the largest magnitude that may be selected for an eigenvalue λ_2 with argument $\theta = \arg[\lambda_2]$ (refer to table I). The maps were derived from the respective FP matrices using algorithm 1. The domain of each map was selected as the unit interval. A positive slope was selected for each linear function defined

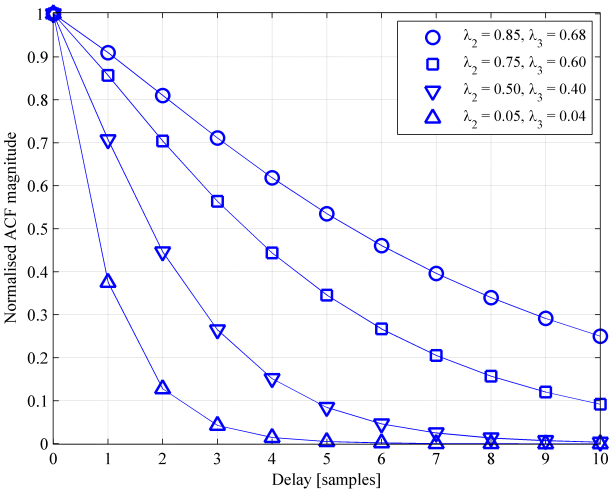

Fig. 4. Measured ACFs generated using the ergodic maps of set $\mathcal{W}_2$.

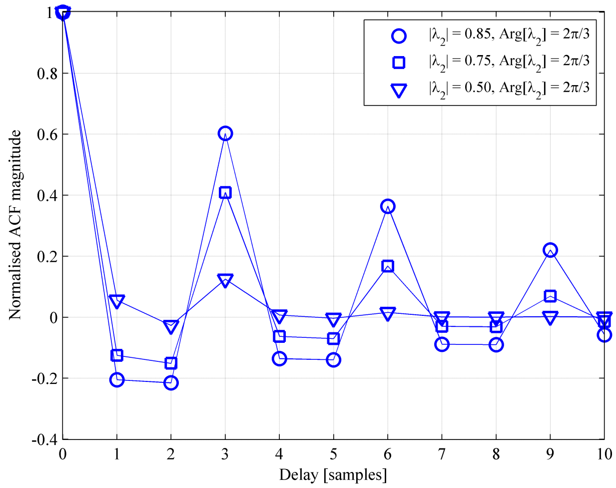

Fig. 6. Measured ACFs generated using the ergodic maps of set $\mathcal{W}_3$.

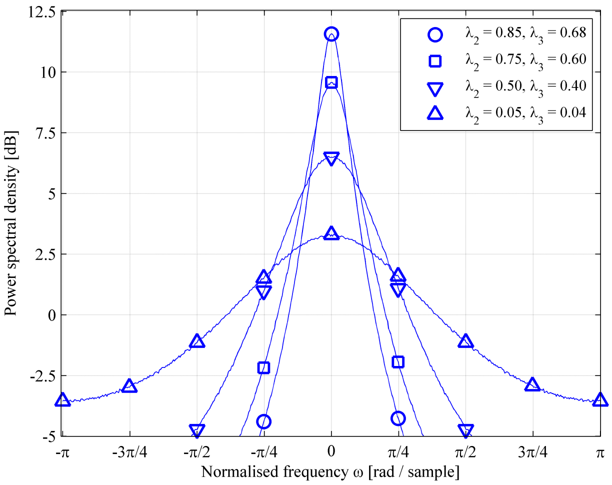

Fig. 5. Measured PSDs generated using the ergodic maps of set $\mathcal{W}_2$.

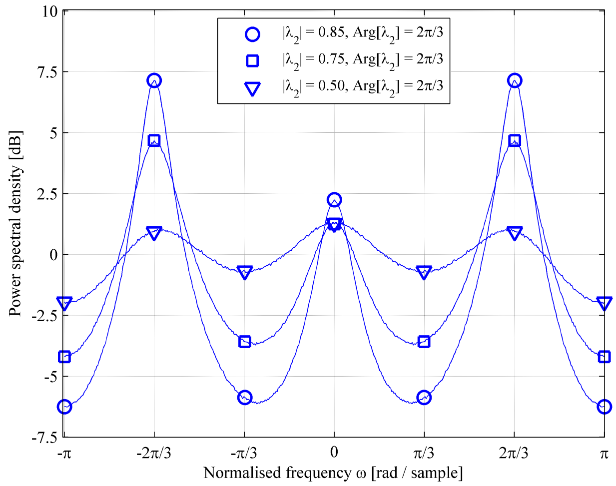

Fig. 7. Measured PSDs generated using the ergodic maps of set $\mathcal{W}_3$.

over its corresponding segment.

The measured ACFs and PSDs associated with the maps of set $\mathcal{W}_1$ are provided in figs. 2 and 3. In the cases where $\arg[\lambda_2] = \pi/2$ and $\arg[\lambda_2] = 2\pi/3$, the figures reveal the emergence of spectral components with frequencies that correspond to the arguments of the eigenvalues λ_2 and λ_3. For the case where $\arg[\lambda_2] = \pi/3$, it is observed that the largest magnitude that may be selected for the eigenvalues λ_2 and λ_3 is relatively small (specifically, $r_{\max}(\pi/3) = 0.5$). This limitation restricts the minimum bandwidth of these ACF components, with the result that these components are not visually distinguishable from a plot of the PSD. The ACF corresponding to the case of real eigenvalues ($\lambda_2 = \lambda_3 = 0.85$) decays monotonically at a relatively slow rate, which corresponds to a ACF component with a narrow bandwidth around zero frequency.

B. Eigenvalue Magnitude and Bandwidth

In order to characterise the relationship between the magnitude of an eigenvalue and the bandwidth of its corresponding ACF component, two sets $\mathcal{W}_2$ and $\mathcal{W}_3$ of ergodic semi–Markov maps were derived. The first set $\mathcal{W}_2$ consists of maps associated with real FP matrix eigenvalues, whereas the maps of the second set $\mathcal{W}_3$ are associated with strictly complex FP matrix eigenvalues. The two map sets $\mathcal{W}_2$ and $\mathcal{W}_3$ were used to investigate the bandwidths of ACF components centered around zero frequency and a frequency of $2\pi/3$ radians per sample, respectively. The selection of FP matrix eigenvalues for these sets are summarised in table I. The maps of both sets $\mathcal{W}_2$ and $\mathcal{W}_3$ were derived from the respective FP matrices in the same manner as was done for the maps in the set $\mathcal{W}_1$.

The ACFs and PSDs corresponding to the maps of set $\mathcal{W}_2$ are presented in figs. 4 and 5, respectively. As anticipated from eq. 4, the ACFs corresponding to these maps are monotonically decreasing and the respective PSDs are concentrated around zero frequency. An inverse relationship is observed between the bandwidth of the ACF component and the magnitude of the eigenvalues λ_2 and λ_3. This observation is supported from fig. 4, which reveals an inverse relationship between the rate of decay of the ACF and the eigenvalue magnitudes.

Figs. 6 and 7 present the ACFs and PSDs associated with the maps of set $\mathcal{W}_3$. A similar inverse relationship is observed between the bandwidth of the signal components centered around $\pm 2\pi/3$ radians per sample and the magnitude of the corresponding eigenvalues.

VI. Summary and Conclusions

A novel technique for constructing semi–Markov ergodic maps was proposed in this paper. The technique uses a new solution to the inverse eigenvalue problem to construct 3–by–3 doubly stochastic FP matrices with complex eigenvalues. These eigenvalues were shown to govern, in part, the characteristics of the resulting map's ACF and PSD. It was demonstrated experimentally that distinct spectral characteristics may be realised through appropriate selection of the FP matrix eigenvalues. Furthermore, it was proved that the proposed solution to the inverse eigenvalue problem is able to realise all possible eigenvalue triples for doubly stochastic matrices, thereby providing greater freedom in selecting the power spectrum. By facilitating the selection of complex eigenvalues, the proposed technique allows for the selection of the oscillation frequency of ACF components. This element of novelty improves the flexibility and versatility of the proposed technique, as compared to existing techniques [15].

It is noted that the proposed technique is limited to producing ergodic maps with a uniform invariant distribution. However, it is anticipated that the FP matrices that comprise the proposed solution to the inverse eigenvalue problem may be used to construct semi–Markov chaotic maps with larger FP matrices and invariant densities that approximate a prescribed density, while reserving greater flexibility with regards to the ACF. The generalisation of the recursive Markov state disaggregation technique, and its application to the solution of the inverse FP problem for both a prescribed invariant density and ACF, is a topic for future research.

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References

[1] E. Ott, *Chaos in dynamical systems*. 2nd ed., Cambridge University Press, Cambridge, 2002.

[2] M. A. Van Wyk and W.-H. Steeb, *Chaos in electronics*. 1st ed., Springer Science & Business Media, Dordrecht, 1997.

[3] M. P. Shaw, V. V. Mitin, E. Schöll and H. L. Grubin, *The physics of instabilities in solid state electron devices*. 1st ed., Springer Science & Business Media, New York, 1992.

[4] F. T. Arecchi, W. Gadomski, and R. Meucci, "Generation of chaotic dynamics by feedback on a laser," *Physical Review A*, vol. 34, no. 2, pp. 1617–1620, 1986.

[5] M. Wiercigroch and B. De Kraker, *Applied nonlinear dynamics and chaos of mechanical systems with discontinuities*. World Scientific series on nonlinear science, series A, vol. 28, World Scientific Publishing, Singapore, 2000.

[6] S. H. Strogatz, *Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering*. 2nd ed., Westview Press, Boulder, 2014.

[7] A. Medio and G. Gallo, *Chaotic dynamics: Theory and applications to economics*. 1st ed., Cambridge University Press, Cambridge, 1992.

[8] M. Casdagli, "Nonlinear prediction of chaotic time series," *Physica D: Nonlinear Phenomena*, vol. 35, no. 3, pp. 335–356, 1989.

[9] S. N. Wood, "Statistical inference for noisy nonlinear ecological dynamic systems," *Nature*, vol. 466, no. 7310, pp. 1102–1104, 2010.

[10] E. M. Bollt, "Controlling chaos and the inverse Frobenius–Perron problem: Global stabilization of arbitrary invariant measures," *International Journal of Bifurcation and Chaos*, vol. 10, no. 05, pp. 1033–1050, 2000.

[11] F. K. Diakonos, D. Pingel, and P. Schmelcher, "A stochastic approach to the construction of one–dimensional chaotic maps with prescribed statistical properties," *Physics Letters A*, vol. 264, no. 2, pp. 162–170, 1999.

[12] A. Baranovsky and D. Daems, "Design of one–dimensional chaotic maps with prescribed statistical properties," *International Journal of Bifurcation and Chaos*, vol. 5, no. 06, pp. 1585–1598, 1995.

[13] X. Nie and D. Coca, "A new approach to solving the inverse Frobenius–Perron problem," in *Proc. European Control Conference (ECC),* 2013, pp. 2916–2920.

[14] ——, "Reconstruction of one–dimensional chaotic maps from sequences of probability density functions," *Nonlinear Dynamics*, vol. 80, no. 3, pp. 1373–1390, 2015.

[15] A. McDonald and M. van Wyk, "Solution of the inverse Frobenius–Perron problem for semi–Markov chaotic maps via recursive Markov state disaggregation," in *Proc. 25th European Signal Processing Conference (EUSIPCO),* 2017, pp. 1654–1658.

[16] H. Mori, B.-C. So, and T. Ose, "Time–correlation functions of one–dimensional transformations," *Progress of Theoretical Physics*, vol. 66, no. 4, pp. 1266–1283, 1981.

[17] A. Rogers, R. Shorten, D. M. Heffernan, and D. Naughton, "Synthesis of piecewise–linear chaotic maps: Invariant densities, autocorrelations, and switching," *International Journal of Bifurcation and Chaos*, vol. 18, no. 08, pp. 2169–2189, 2008.

[18] L. Ciampolini, S. Meignen, O. Menut, and T. David, "Direct solution of the inverse stochastic problem through elementary Markov state disaggregation," Tech. Rep. HAL–01016804, HAL Archives, 2014.

[19] A. Lasota and M. C. Mackey, *Chaos, fractals, and noise: Stochastic aspects of dynamics*. 2nd ed., Springer Applied Mathematical Sciences vol. 97, New York, 1994.

[20] P. Góra and A. Boyarsky, "A matrix solution to the inverse Perron–Frobenius problem," *Proceedings of the American Mathematical Society*, vol. 118, no. 2, pp. 409–414, 1993.

[21] H. Perfect and L. Mirsky, "Spectral properties of doubly–stochastic matrices," *Monatshefte für Mathematik*, vol. 69, no. 1, pp. 35–57, 1965.

[22] W. T. Bayha, "The generalized rotation matrix and the eigenvalue problem," *American Journal of Physics*, vol. 52, no. 4, pp. 370–371, 1984.